Bending Deflection

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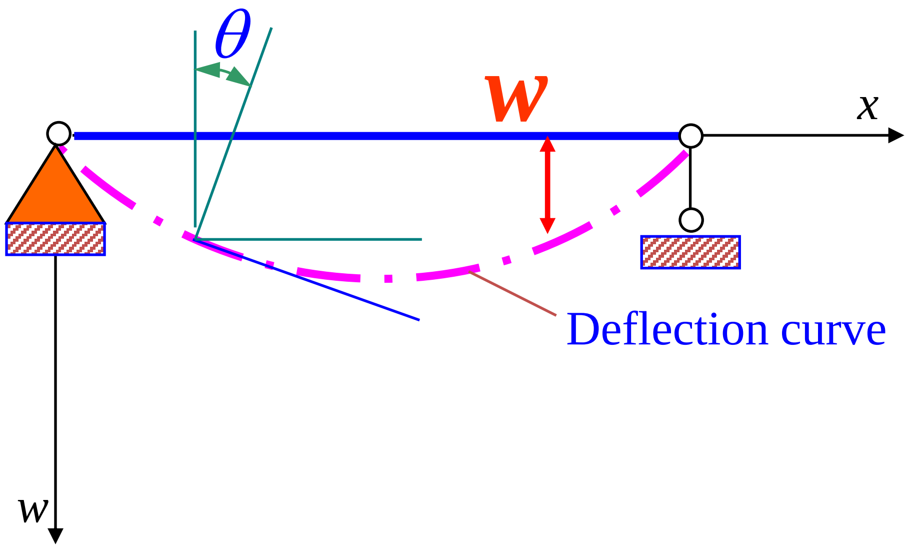
The Elastic Curve, Deflection and Slope

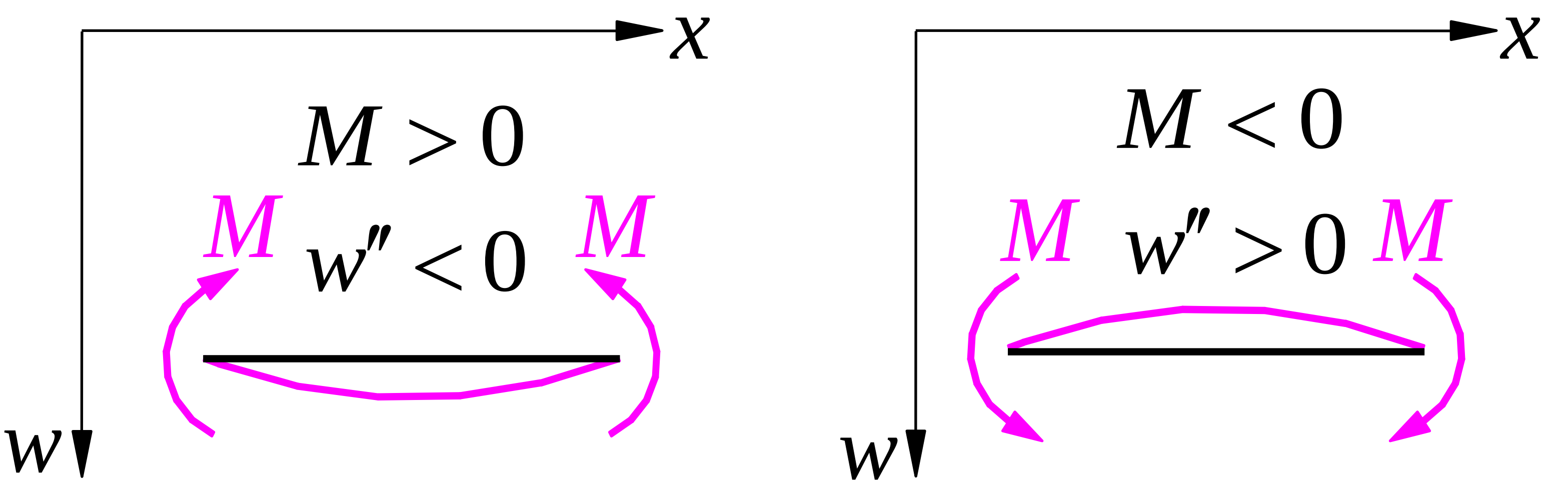
- **The elastic curve:** beam axis under bending, required to determine beam deflection and slope.
- **Bending deflections** ($w = f(x)$): vertical deflection of the neutral surface, defined as **downward positive / upward negative.**
- **Slope** ($\theta = \theta(x) \approx \tan(\theta) = dw/dx$): rotation of cross-sections, defined as **clockwise positive / counter clockwise negative**



Differential Equation of the Elastic Curve

- Curvature of the neutral surface

$$\frac{1}{\rho(x)} = \frac{M(x)}{EI_z} \qquad \kappa = \frac{1}{\rho(x)} = -\frac{w''}{(1+w'^2)^{3/2}} \approx -w''$$



$$EIw'' = -M$$

EI: flexural rigidity

- The negative sign is due to the particular choice of the *w*-axis.

Deflection and Slope by Integration

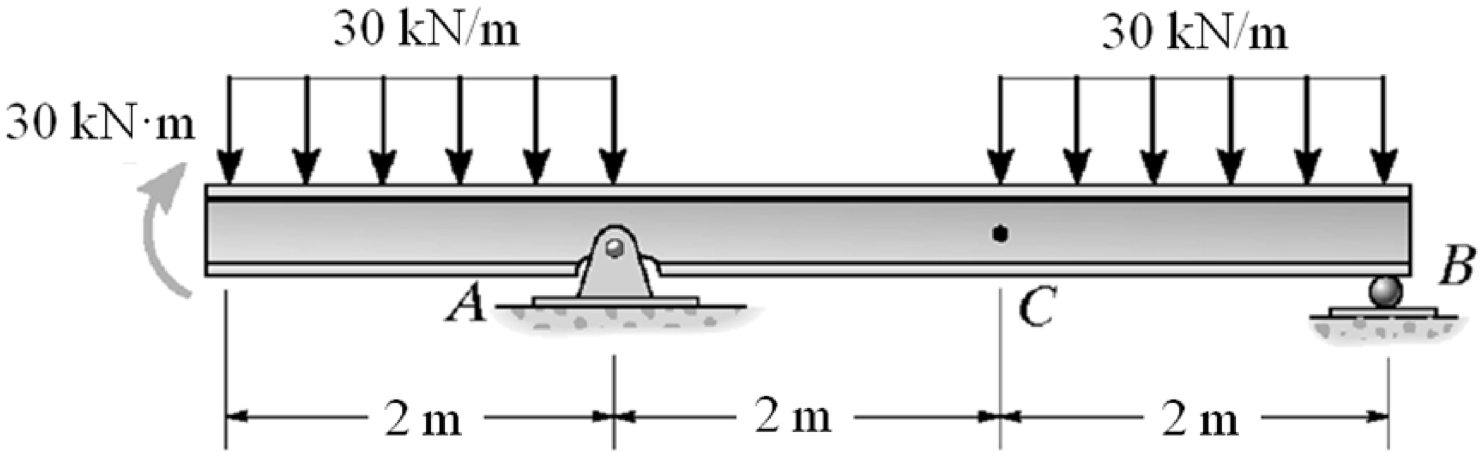
$$EIw'' = -M(x)$$

$$EIw' = -\int M(x)\mathrm{d}x + C$$

$$EIw = -\int\int M(x)\mathrm{d}x\mathrm{d}x + Cx + D$$

- Conventionally assuming constant flexural rigidity (*EI*)
- Integration constants *C* and *D* can be determined from boundary conditions, symmetry conditions, and continuity conditions.

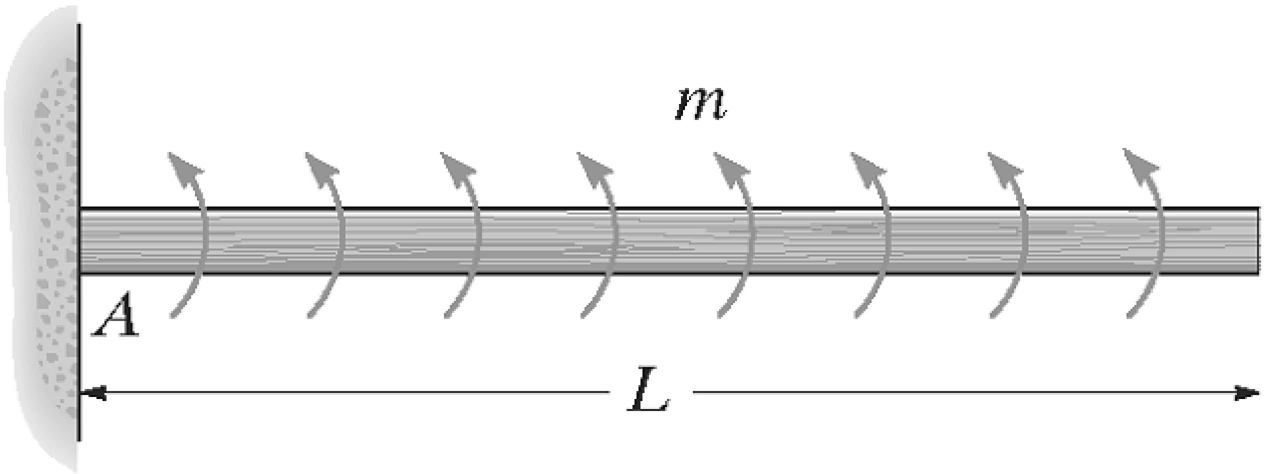
Boundary Conditions – Simple Beams



- Deflections are restrained at the hinged/rolled supports

$$\Rightarrow w_A = 0; \quad w_B = 0$$

Boundary Conditions- Cantilever Beams



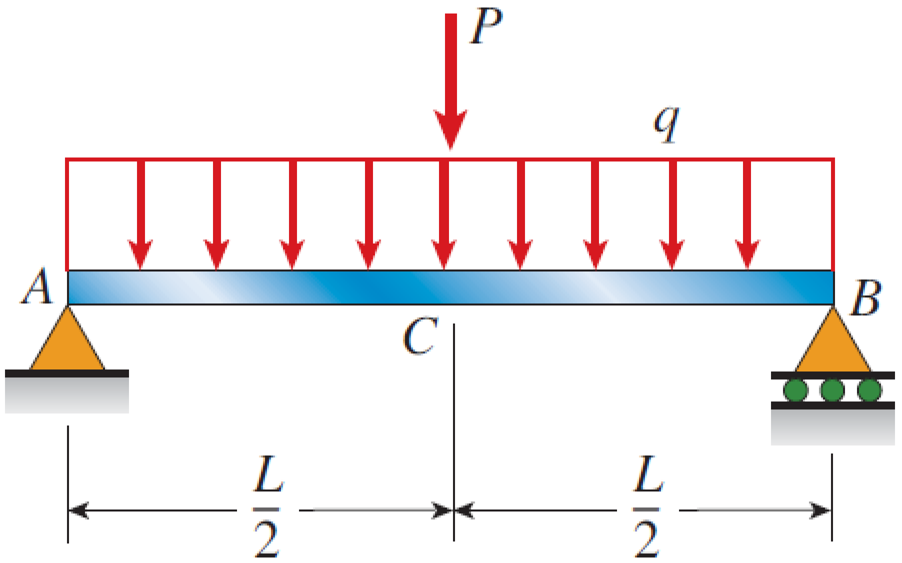
- Both the deflection and rotation are restrained at the clamped end

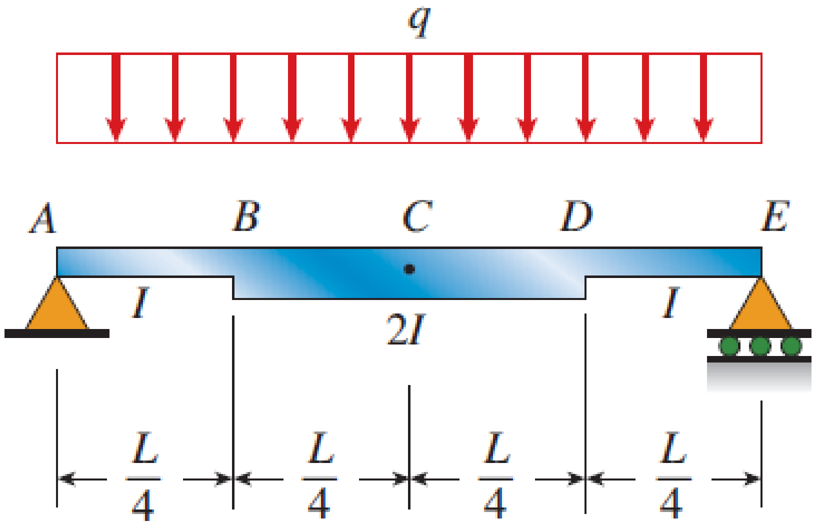
$$\Rightarrow w_A = 0; \quad \theta_A = 0$$

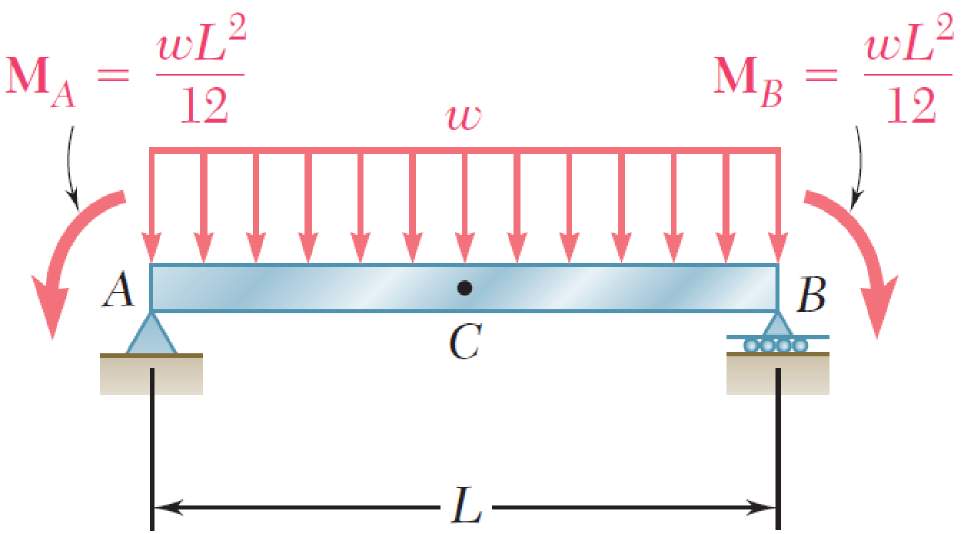
Symmetry Conditions

- Both the geometry and loads are symmetric about the mid-section ($x = L/2$)

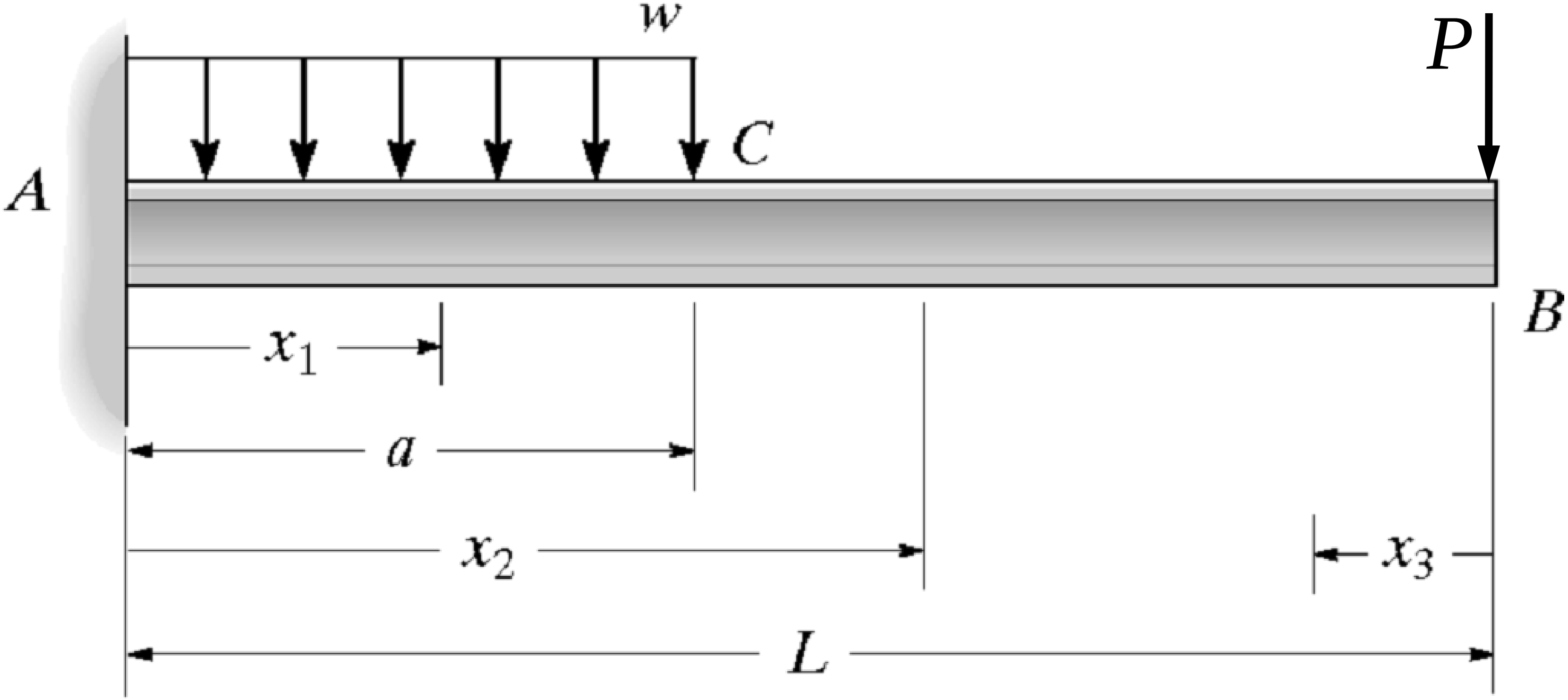
$$\Rightarrow \theta_C = 0$$







Continuity Conditions

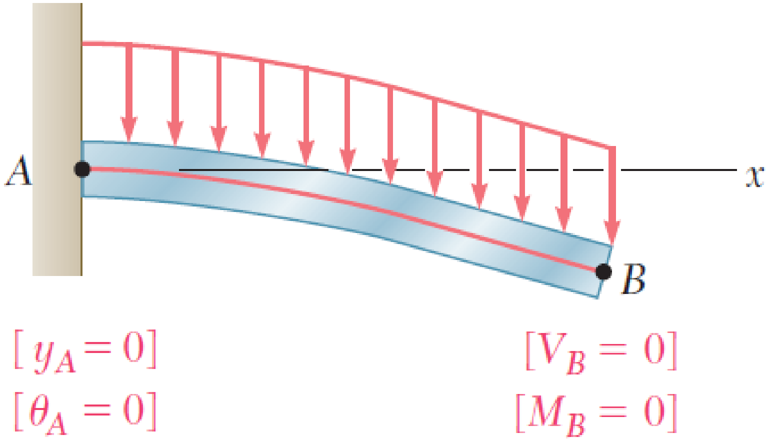


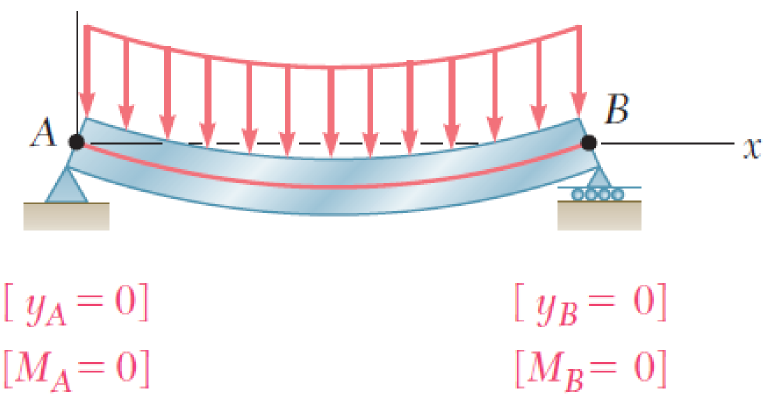
$$0 \le x_1 \le a, \quad a \le x_2 \le L, \quad 0 \le x_3 \le L-a$$

$$w\left(x_1 = a\right) = w\left(x_2 = a\right); \quad \theta\left(x_1 = a\right) = \theta\left(x_2 = a\right)$$

$$w\left(x_1 = a\right) = w\left(x_3 = L-a\right); \quad \theta\left(x_1 = a\right) = -\theta\left(x_3 = L-a\right)$$

Direct Integration from Distributed Loads





- For a beam subjected to distributed loads

$$\frac{dM}{dx} = F_S(x), \qquad \frac{d^2M}{dx^2} = \frac{dF_S}{dx} = q(x)$$

- Equation for beam displacement becomes

$$EI\frac{d^4w}{dx^4} = -\frac{d^2M}{dx^2} = -q(x)$$

- Integrating four times yields

$$EI\,w(x) = -\int dx \int dx \int dx \int q(x)\,dx$$
$$+ \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from conditions on the shear forces and bending moments as well as conditions on the slopes and deflections.

Direct Integration from Transverse Loads

- For a beam subjected to transverse loads (without distributed loads)

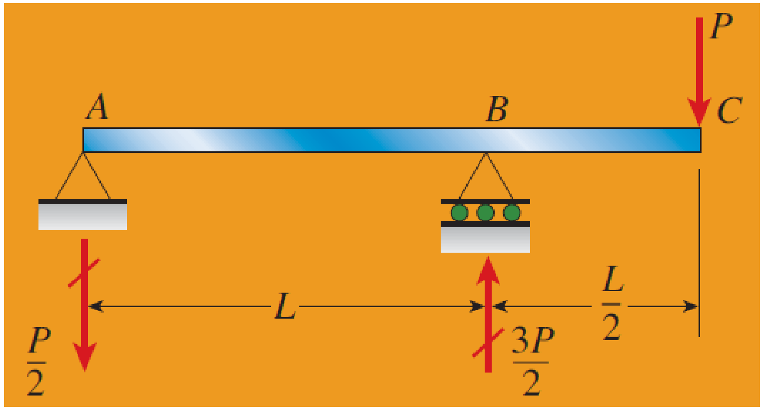
- Equation for beam displacement becomes

$$EI\,\frac{d^3w}{dx^3} = -\frac{dM}{dx} = -F_S(x)$$

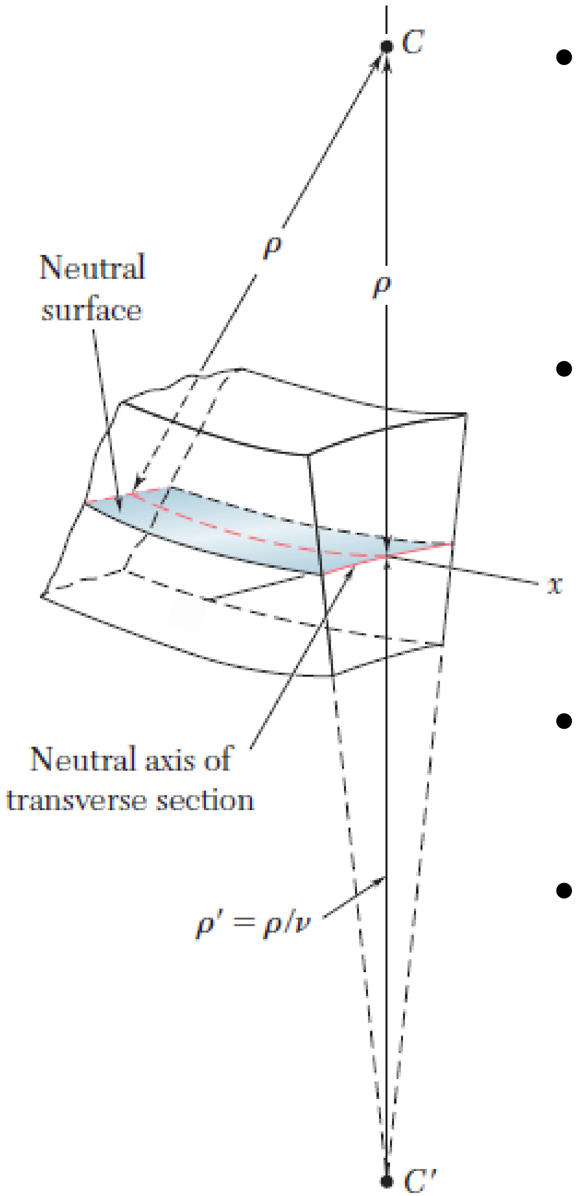
- Integrating three times yields

$$EI\,w(x) = -\int dx \int dx \int F_S(x)\,dx$$
$$+\tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from conditions on the bending moments as well as conditions on the slopes and deflections.



Deformations in a Transverse Cross Section



- Deformation due to bending moment is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_x(y)}{y} = \frac{\sigma_x(y)}{Ey}$$

- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

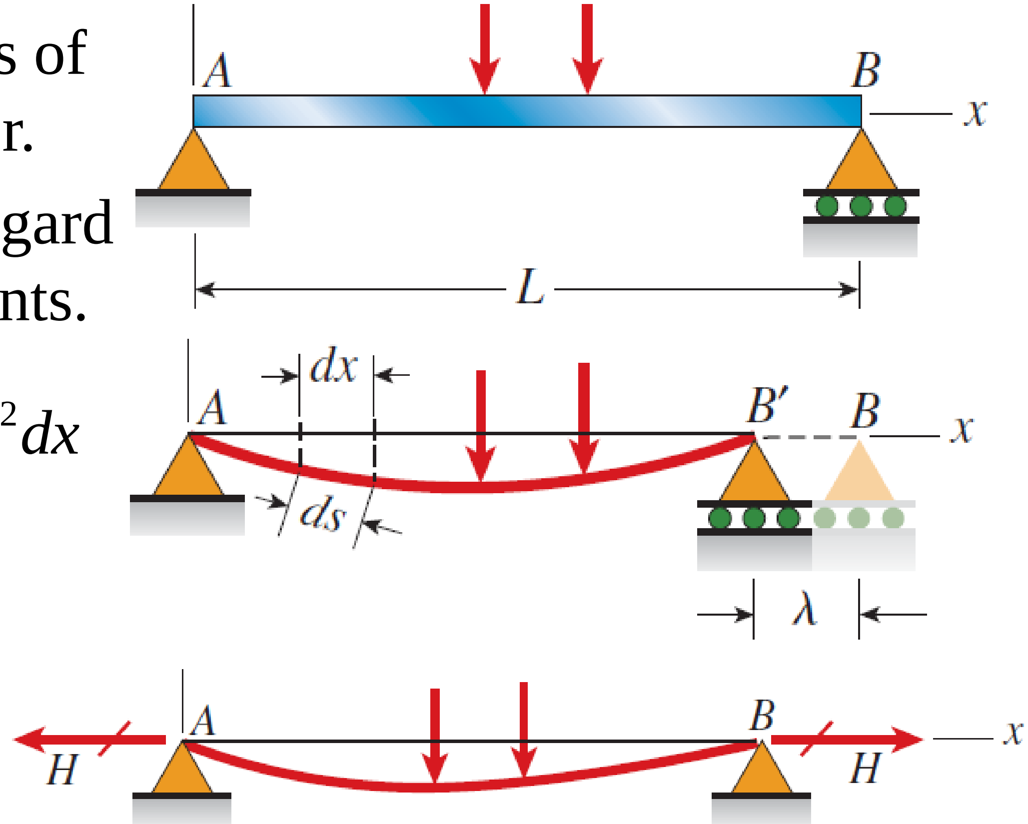
$$\varepsilon_y(y) = -\nu\varepsilon_x(y) = -\frac{\nu y}{\rho}, \quad \varepsilon_z(y) = -\nu\varepsilon_x(y) = -\frac{\nu y}{\rho}$$

- For a rectangular cross-section, no change in the vertical dimension will be observed.
- Horizontal expansion above the neutral surface and contraction below it cause an in-plane curvature

$$\frac{1}{\rho'} = -\frac{\varepsilon_z(y)}{y} = \frac{\nu}{\rho} = \text{ anticlastic curvature}$$

Curvature Shortening

- When a beam is bent, the ends of the beam move closer together.
- It is common practice to disregard these longitudinal displacements.



$$ds - dx = \left(\sqrt{1+w'^2} - 1\right)dx \approx \frac{1}{2}w'^2 dx$$

$$\lambda = L_{AB} - L_{AB'} = \int_0^L \frac{1}{2}w'^2 dx$$

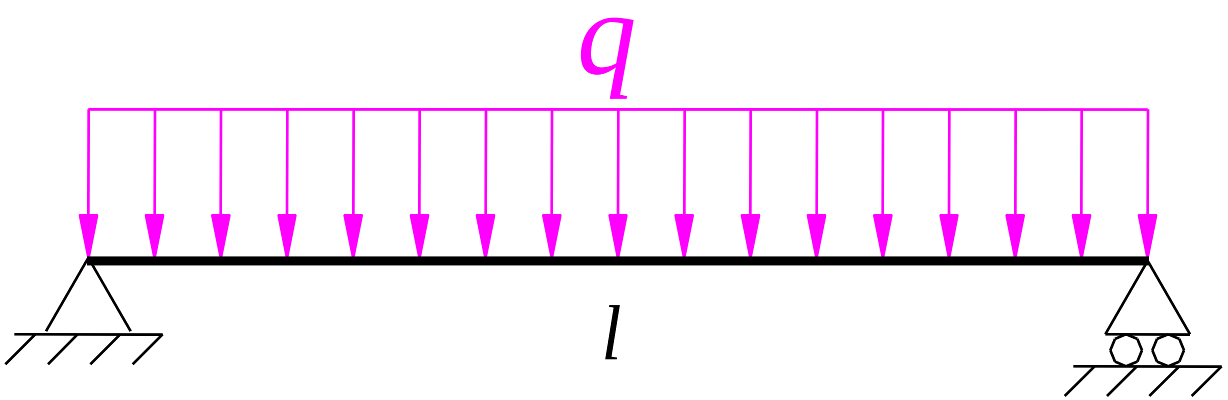
- For immovable supports, a horizontal reaction will develop at each end.

$$\lambda = HL/EA \quad \Rightarrow \quad H = \lambda EA/L$$

- This equation gives *a close estimate* of the tensile stress produced by the immovable supports of a simple beam.

Sample Problem

- Given: flexural rigidity (EI) of a simply supported beam under a uniformly distributed load of density q
- Find: equations of deflections and slopes, and their maximum values ($\theta_{\max}$, $w_{\max}$)



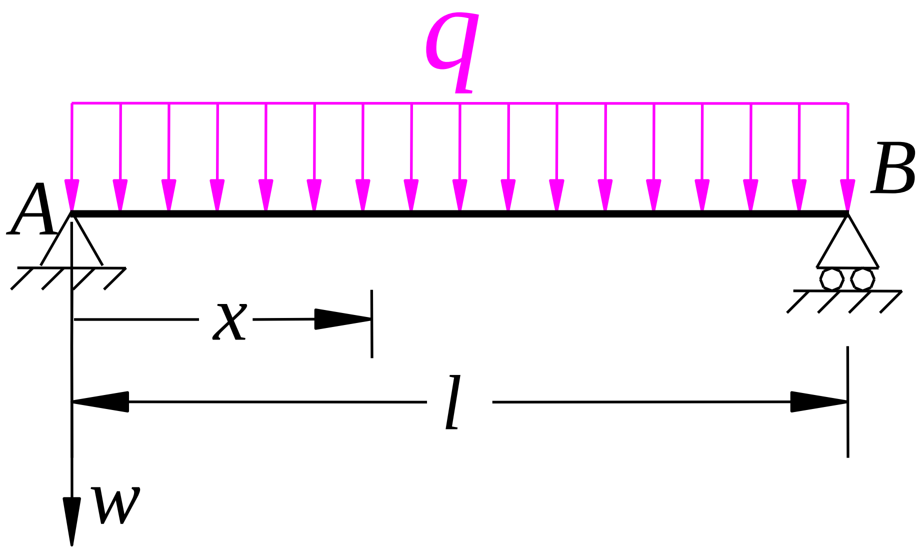
- Solution:

$$M(x) = \frac{ql}{2}x - \frac{q}{2}x^2$$

$$EIw'' = -\frac{ql}{2}x + \frac{q}{2}x^2$$

$$EIw' = -\frac{ql}{4}x^2 + \frac{q}{6}x^3 + C$$

$$EIw = -\frac{ql}{12}x^3 + \frac{q}{24}x^4 + Cx + D$$



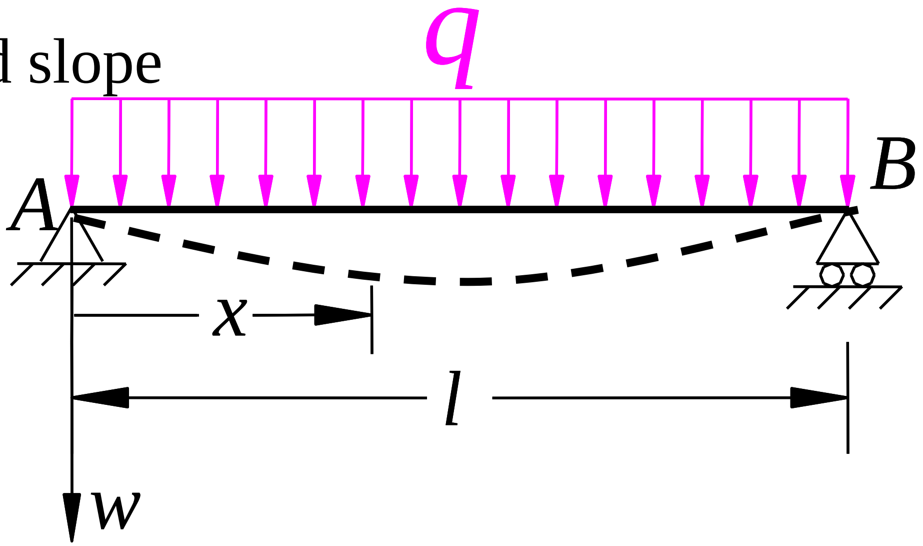
- Boundary conditions: $w\left(x=0\right)=0,\ w\left(x=l\right)=0$

$$\Rightarrow C = \frac{ql^3}{24}, \quad D = 0$$

- Equations of beam deflection and slope

$$\theta = \omega' = \frac{q}{24EI}(l^3 - 6lx^2 + 4x^3)$$

$$w = \frac{qx}{24EI}(l^3 - 2lx^2 + x^3)$$



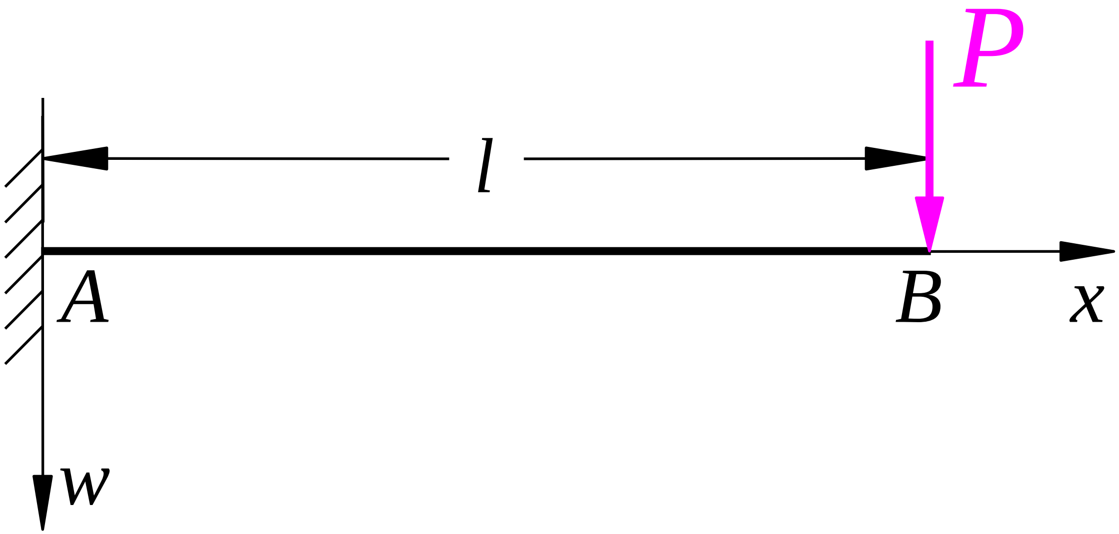
- The maximum deflection and slope

$$\theta_{\max} = \theta_A = -\theta_B = \frac{ql^3}{24EI}$$

$$w_{\max} = w\left(x = \frac{l}{2}\right) = \frac{5ql^4}{384EI}$$

Sample Problem

- Given: flexural rigidity (EI) of a cantilever beam under a concentrated load acting at its free end
- Find: equations of deflections and slopes, and their maximum values (θ_{max}, w_{max})



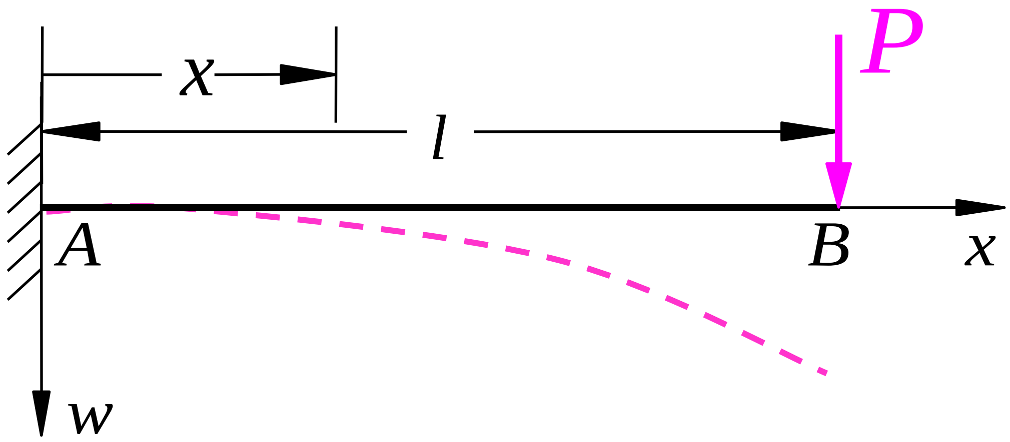
- Solution:

$$M(x) = -P(l-x)$$

$$EIw'' = -Px + Pl$$

$$EIw' = -\frac{P}{2}x^2 + Plx + C$$

$$EIw = -\frac{P}{6}x^3 + \frac{Pl}{2}x^2 + Cx + D$$



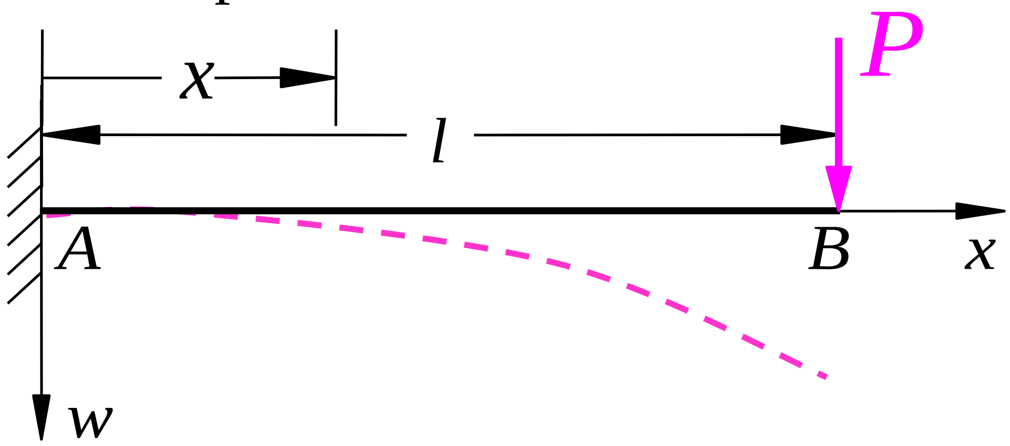
- Boundary conditions: $w\left(x=0\right)=0, \quad w'\left(x=0\right)=0$

$$\Rightarrow C = D = 0$$

- Equations of beam deflection and slope

$$\theta = \frac{Px}{2EI}(2l - x)$$

$$w = \frac{Px^2}{6EI}(3l - x)$$



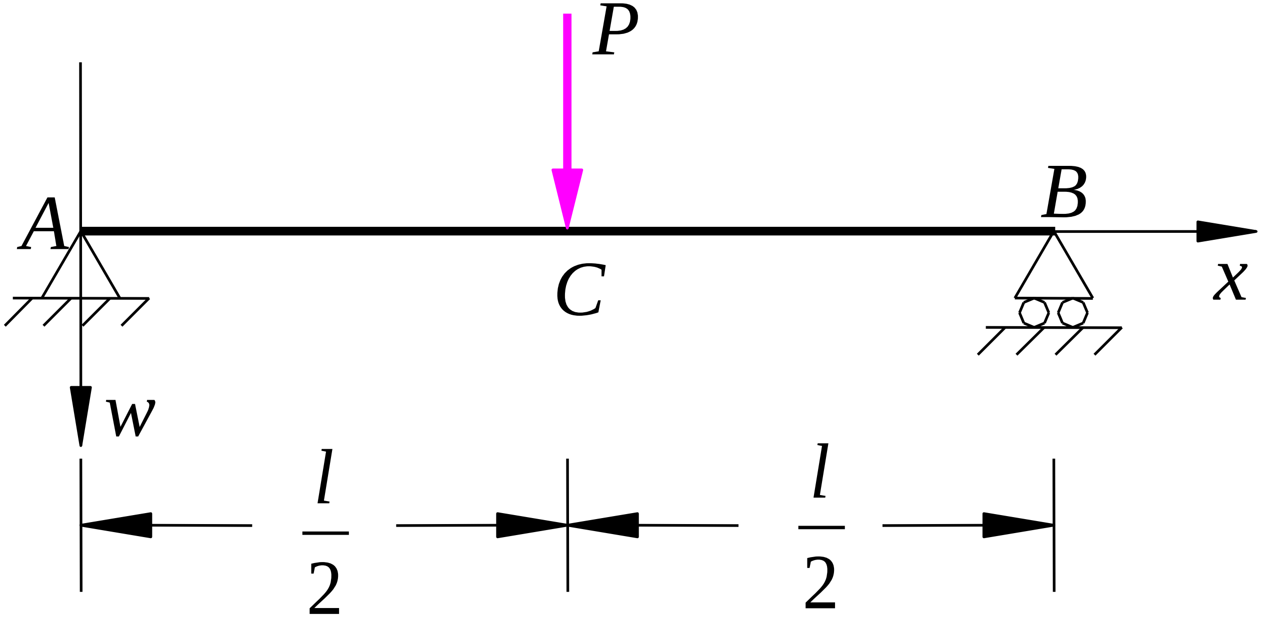
- The maximum deflection and slope

$$\theta_{\max} = \theta_B = \frac{Pl^2}{2EI}$$

$$w_{\max} = w_B = \frac{Pl^3}{3EI}$$

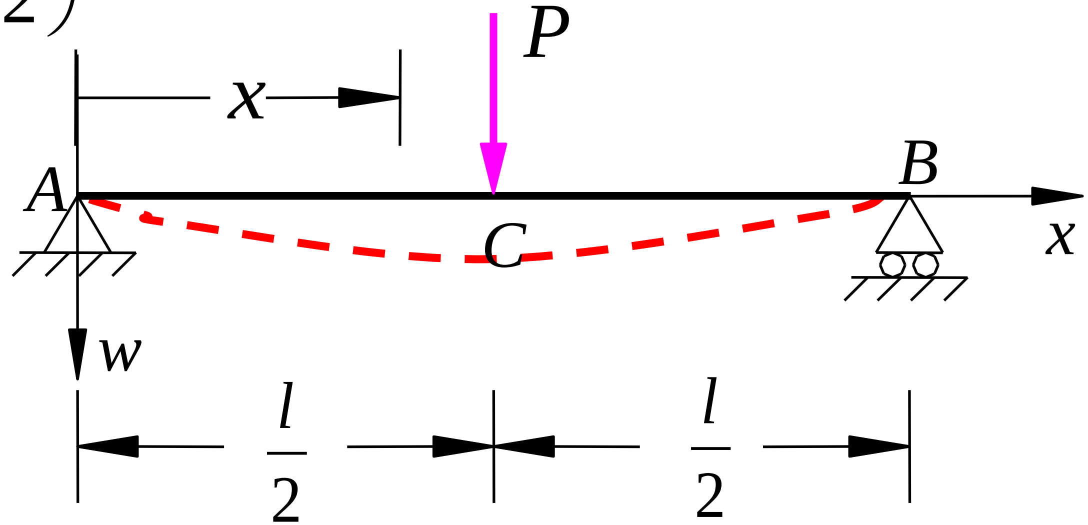
Sample Problem

- Given: a simply supported beam with flexural rigidity EI is subjected to a concentrated load P as shown
- Find: the equations of deflection and slope, and their maximum values (w_{max}, θ_{max})



- Solution
- Because of symmetry, it's sufficient to solve only portion *AC*.

$$M(x) = \frac{P}{2}x, \qquad \left(0 \le x < \frac{l}{2}\right)$$



$$EIw'' = -\frac{P}{2}x$$

$$EIw' = -\frac{P}{4}x^2 + C$$

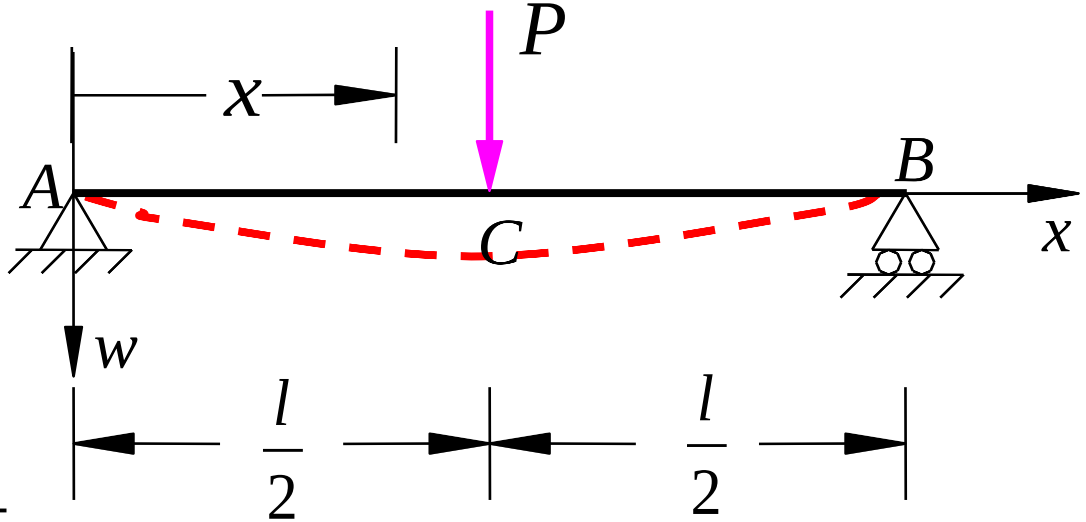
$$EIw = -\frac{P}{12}x^3 + Cx + D$$

- Left boundary condition: $w\left(x=0\right)=0 \quad \Rightarrow D=0$
- Symmetry condition: $w'\left(x=\frac{l}{2}\right)=0 \quad \Rightarrow C=\frac{Pl^2}{16}$

- Equations of bending deflection and slope:

$$\theta = \frac{P}{16EI}(l^2 - 4x^2)$$

$$w = \frac{Px}{48EI}(3l^2 - 4x^2)$$



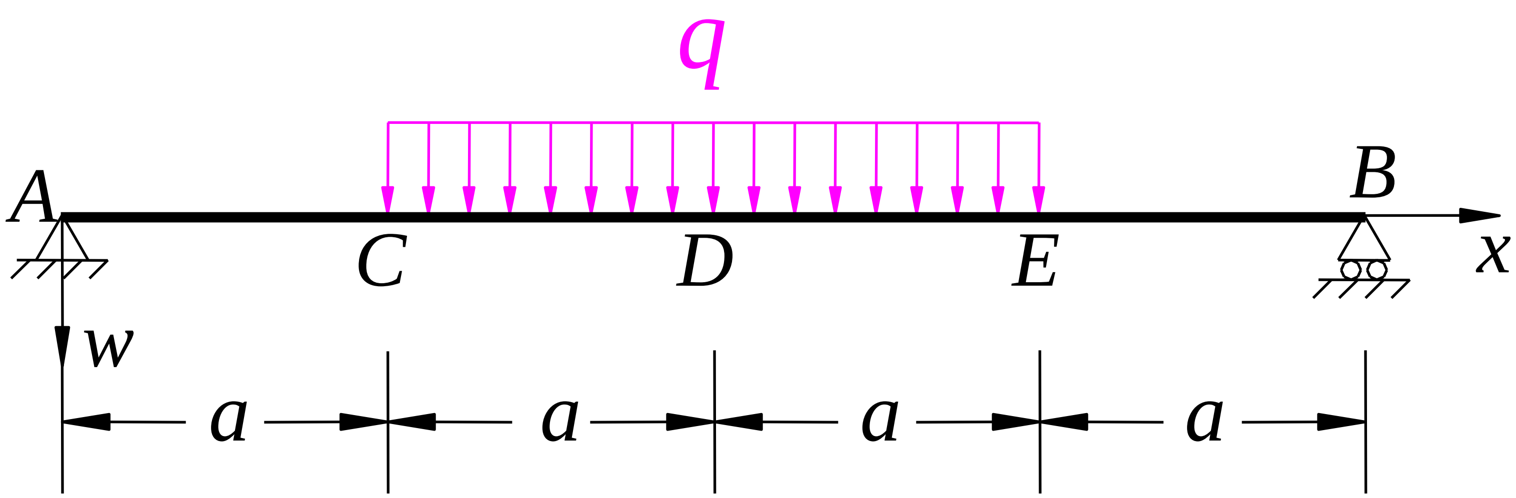
- Maximum deflection and slope:

$$\theta_{\max} = \theta_A = -\theta_B = \frac{Pl^2}{16EI}$$

$$w_{\max} = w\Big|_{x=\frac{l}{2}} = \frac{Pl^3}{48EI}$$

Sample Problem

- Given: a simply supported beam with flexural rigidity EI is subjected to a uniformly distributed load with density q, on its central portion as shown
- Find: the equations of deflection and slope, and their maximum values (w_{max}, θ_{max}).



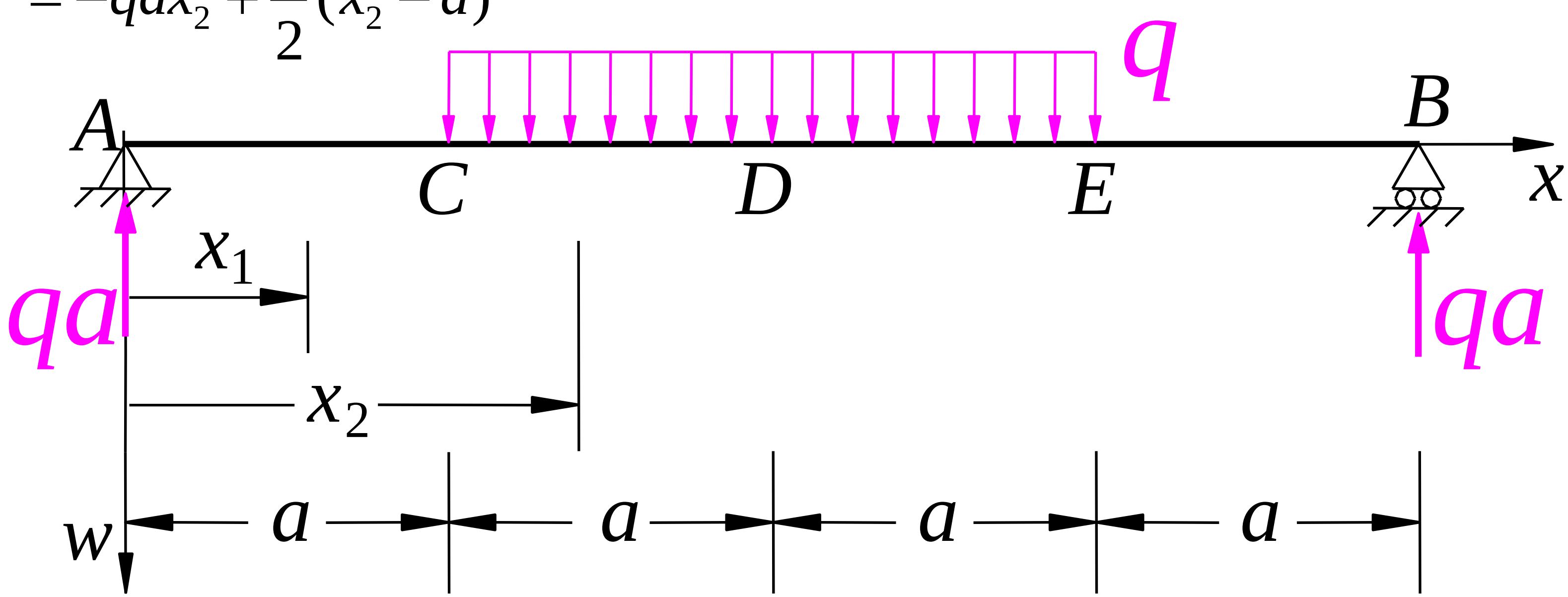
- Solution
- Thanks to symmetry, it is sufficient to consider only the left half

$$M_1(x_1) = qax_1 \qquad (0 \le x_1 \le a)$$

$$M_2(x_2) = qax_2 - \frac{q}{2}(x_2 - a)^2 \qquad (a \le x_2 \le 2a)$$

$$EIw_1'' = -qax_1$$

$$EIw_2'' = -qax_2 + \frac{q}{2}(x_2 - a)^2$$



$$EIw_1'' = -qax_1 \quad \Rightarrow \begin{cases} EIw_1' = -\dfrac{qa}{2}x_1^{\,2} + C_1 \\ EIw_1 = -\dfrac{qa}{6}x_1^{\,3} + C_1 x_1 + D_1 \end{cases}$$

$$EIw_2'' = -qax_2 + \frac{q}{2}(x_2 - a)^2$$

$$\Rightarrow \begin{cases} EIw_2' = -\dfrac{qa}{2}x_2^{\,2} + \dfrac{q}{6}(x_2 - a)^3 + C_2 \\ EIw_2 = -\dfrac{qa}{6}x_2^{\,3} + \dfrac{q}{24}(x_2 - a)^4 + C_2 x_2 + D_2 \end{cases}$$

- Due to symmetry: $w_2'\left(x_2 = 2a\right) = 0 \quad \Rightarrow C_2 = \dfrac{11}{6}qa^3$
- Constraint condition: $w_1\left(x_1 = 0\right) = 0 \quad \Rightarrow D_1 = 0$
- Continuity conditions:

$$w_1'\left(x_1 = a\right) = w_2'\left(x_2 = a\right) \quad \Rightarrow C_1 = C_2,\ \ D_1 = D_2$$

- Equations of deflection and slope:

$$\theta_1 = \frac{qa}{6EI}(11a^2 - 3x_1^{\ 2}) \qquad 0 \le x_1 \le a$$

$$\theta_2 = \frac{q}{6EI}[-3ax_2^{\ 2} + (x_2 - a)^3 + 11a^3] \qquad a \le x_2 \le 2a$$

$$w_1 = \frac{qa}{6EI}(11a^2 x_1 - x_1^{\ 3}) \qquad 0 \le x_1 \le a$$

$$w_2 = \frac{q}{24EI}[-4ax_2^{\ 3} + (x_2 - a)^4 + 44a^3 x_2] \qquad a \le x_2 \le 2a$$

- Maximum deflection and slope:

$$\theta_{\max} = \theta_A = \theta_1\Big|_{x_1=0} = \frac{11qa^3}{6EI}, \quad w_{\max} = w_2\Big|_{x_2=2a} = \frac{19qa^4}{8EI}$$

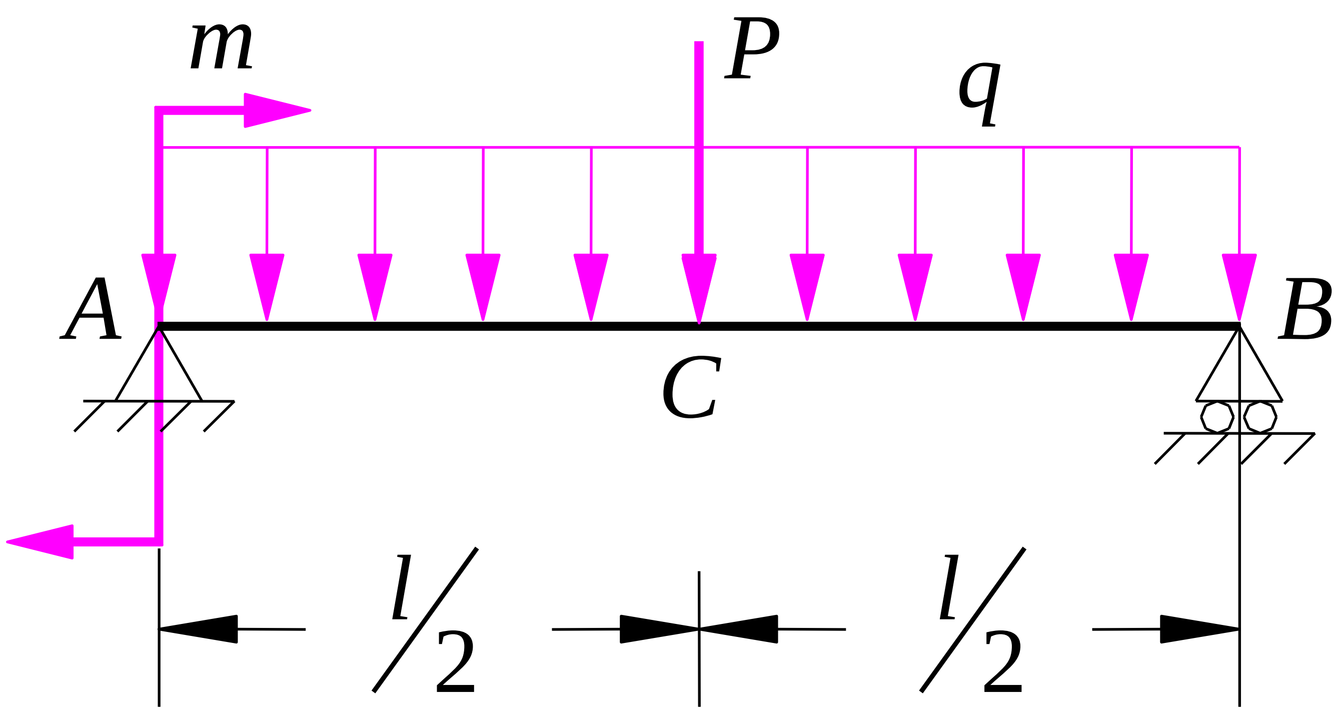
Deflection and Slope by Superposition

- **Superposition of Loads:**
- Deformation of beams subjected to combinations of loads may be obtained as the linear combination of the deformations due to individual loads.

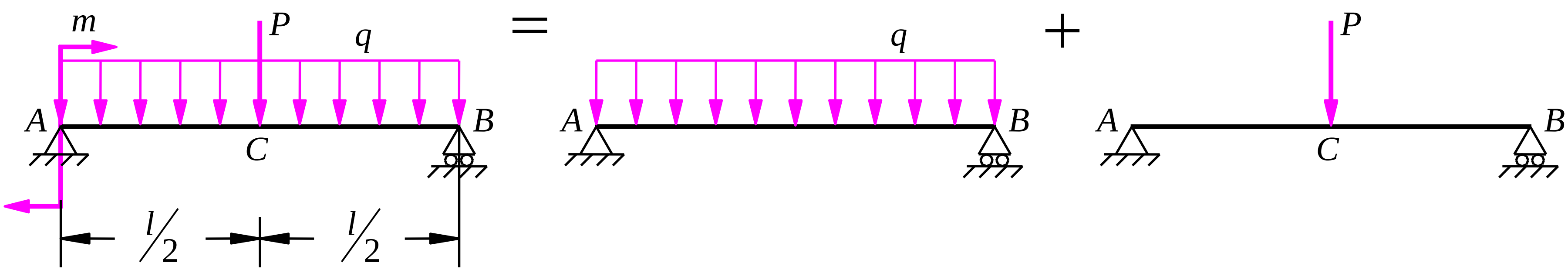
  - Beam material obeys linearly elastic Hooke's law.
  - No interactions exist among deformations induced by individual loads.
  - Procedure is facilitated by tables of solutions for common types of loadings and supports.

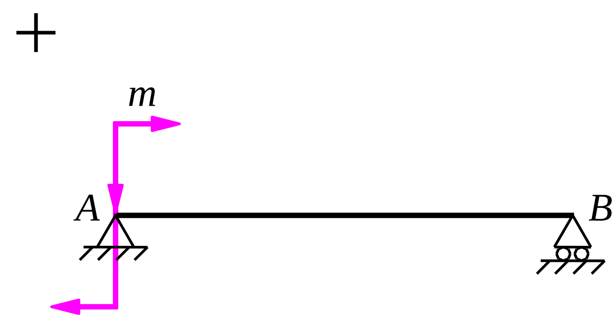
Sample Problem

- Using method of superposition to find the deflection at section *C* and the slopes at sections *A* and *B*.



- Solution:
- Superpose the deformations due to the uniformly distributed load (q), the concentrated load (P) and the concentrated moment (m).





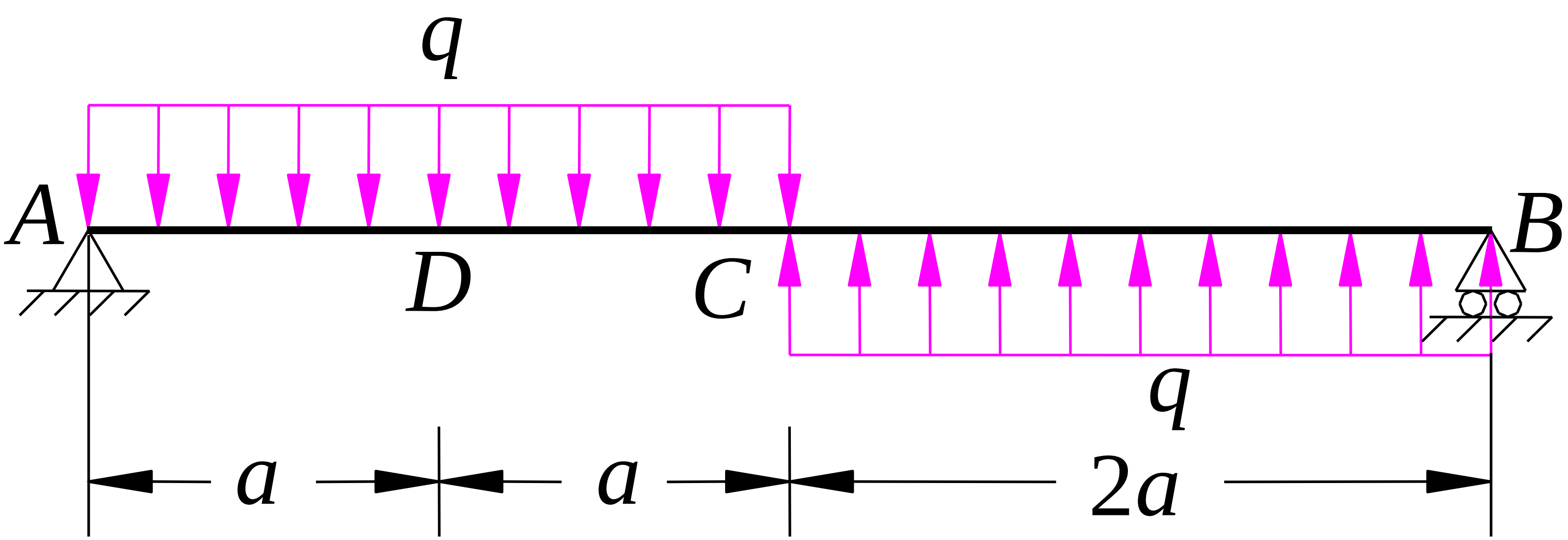
$$w_C = \frac{5ql^4}{384EI} + \frac{Pl^3}{48EI} + \frac{ml^2}{16EI}$$

$$\theta_A = \frac{ql^3}{24EI} + \frac{Pl^2}{16EI} + \frac{ml}{3EI}$$

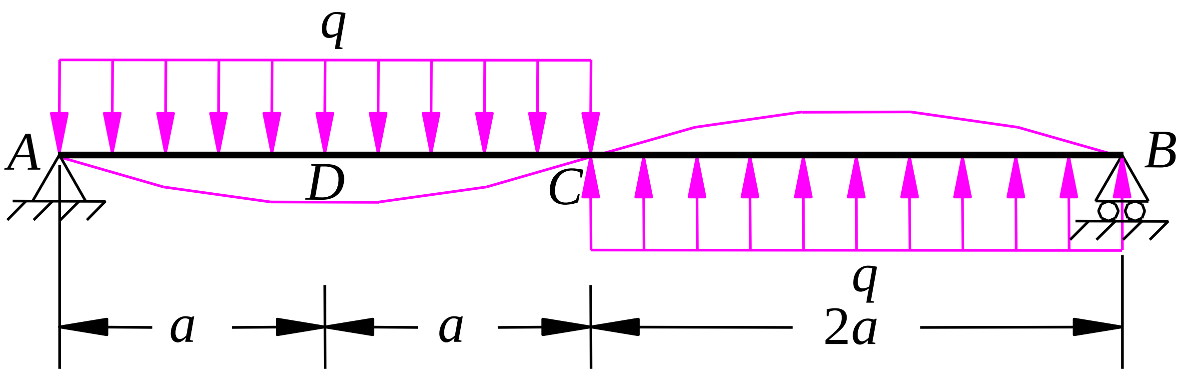
$$\theta_B = -\frac{ql^3}{24EI} - \frac{Pl^2}{16EI} - \frac{ml}{6EI}$$

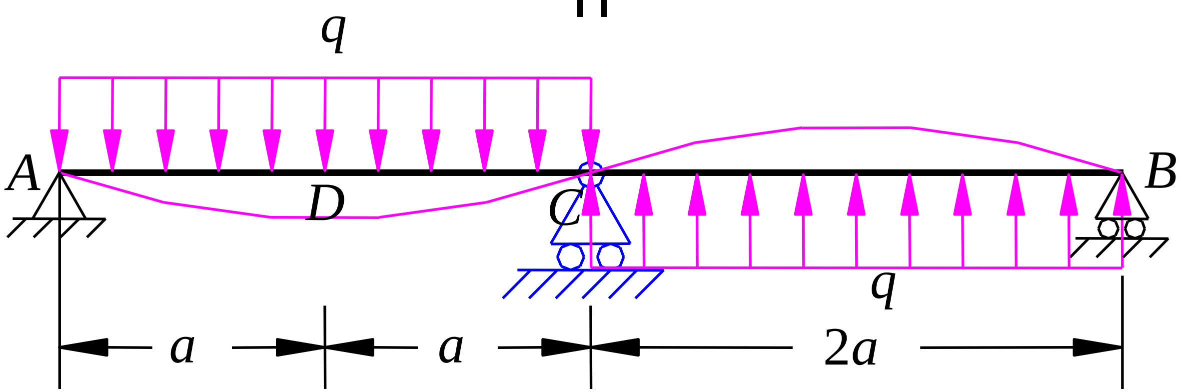
Sample Problem

- Find the deflections at sections *C* and *D*.



- Solution

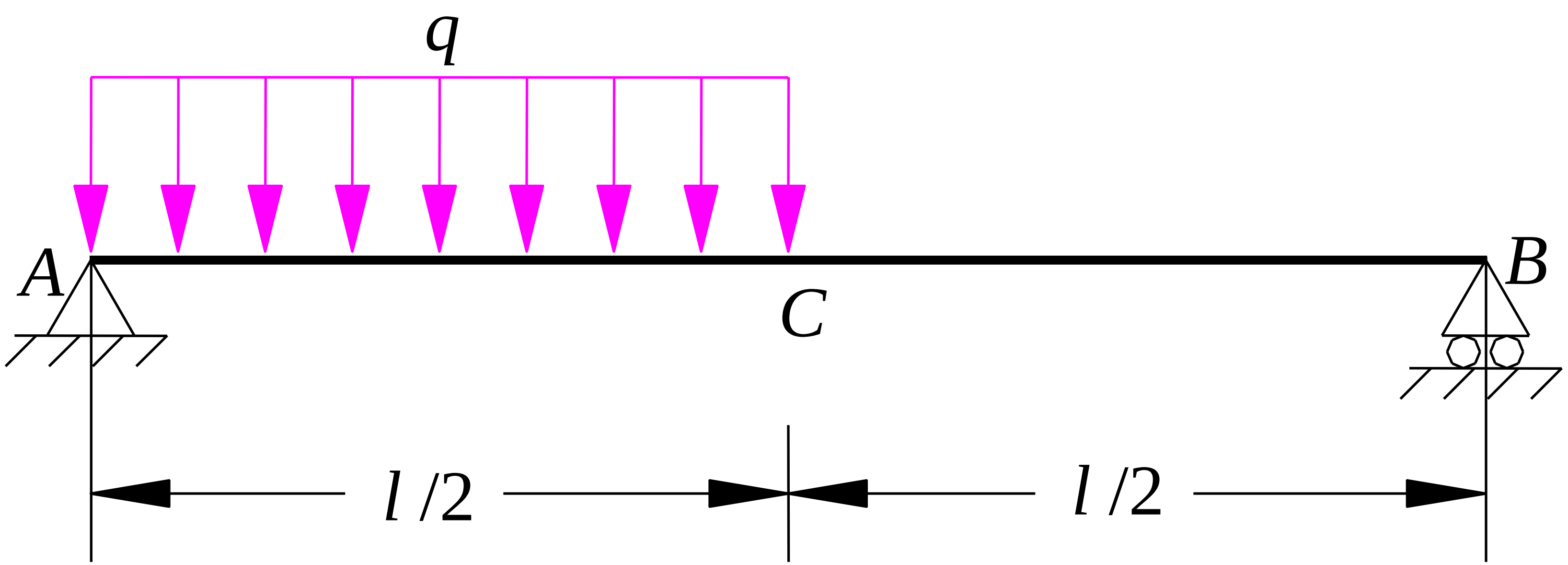




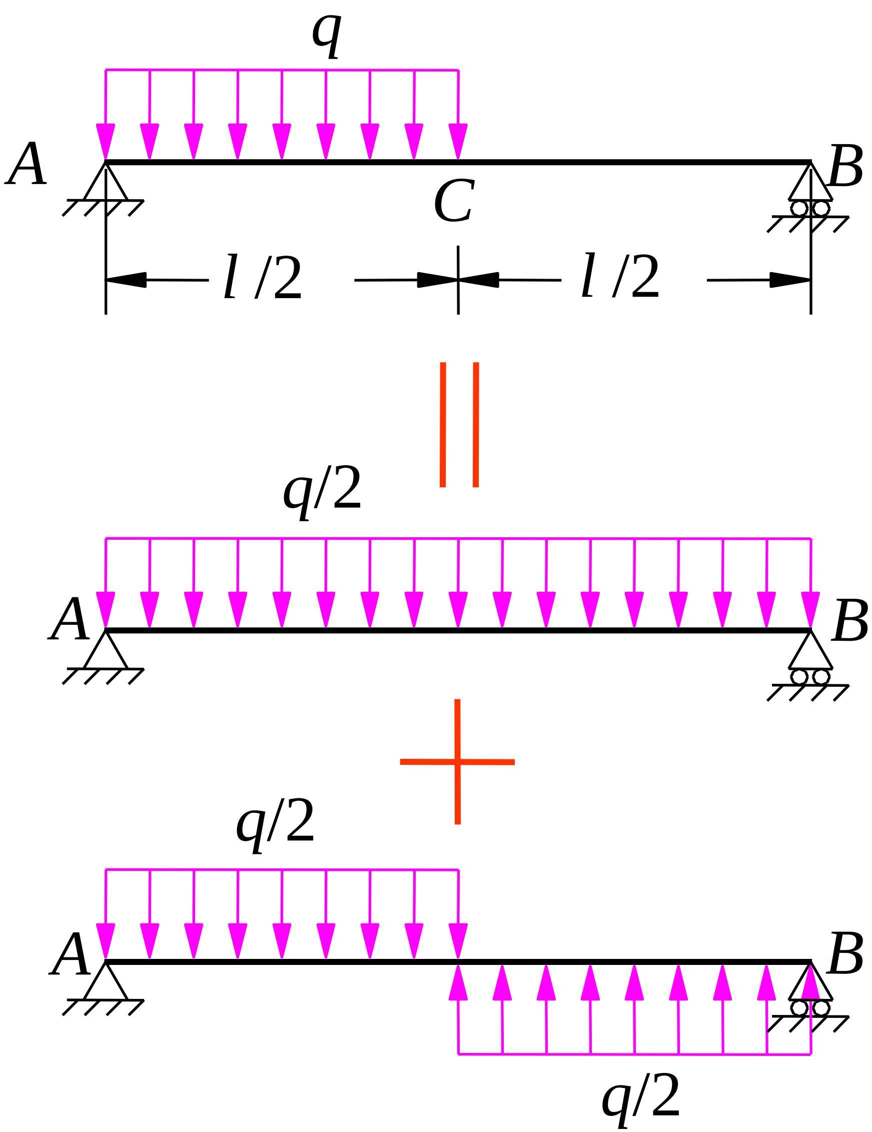
$$w_C = 0, \quad w_D = \frac{5q(2a)^4}{384EI} = \frac{5qa^4}{24EI}$$

Sample Problem

- Find the deflection at section *C* and the slope at section *B*.



- Solution



$$\theta_B = -\frac{\left(\frac{q}{2}\right) l^3}{24EI}$$

$$w_C = w\Big|_{x=\frac{l}{2}} = \frac{5\left(\frac{q}{2}\right) l^4}{384EI}$$

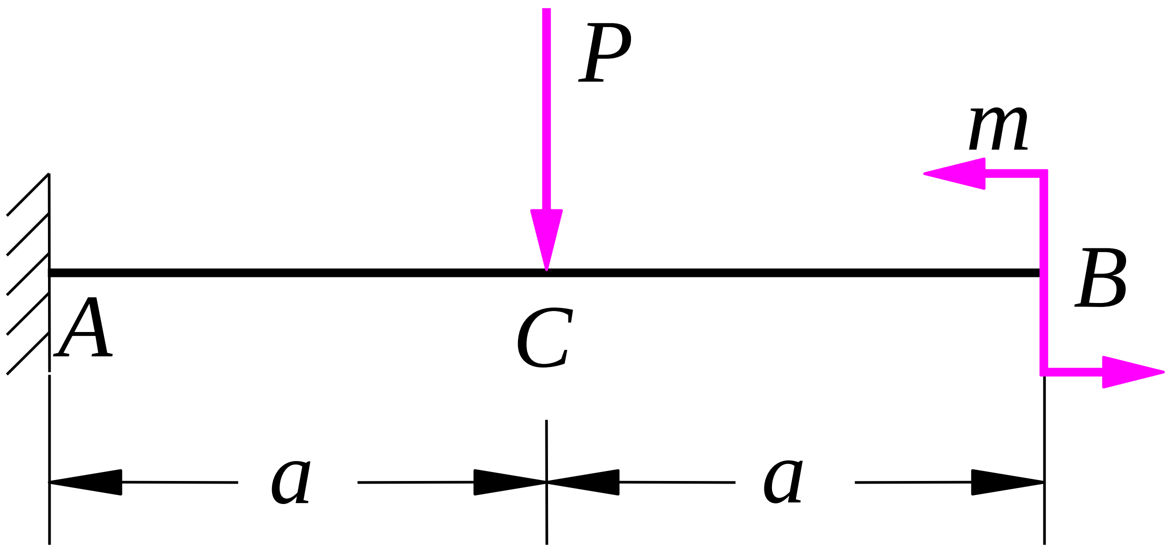
$$+$$

$$\theta_B = -\frac{\left(-\frac{q}{2}\right)\left(\frac{l}{2}\right)^3}{24EI}$$

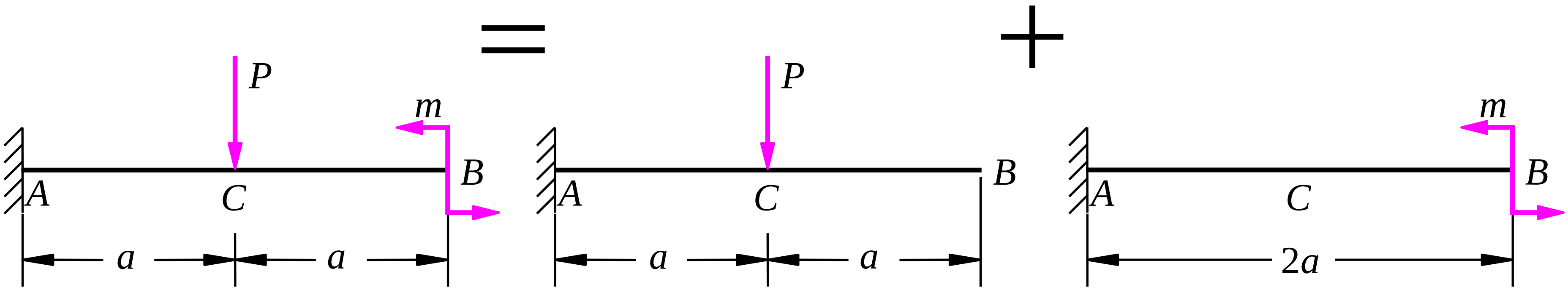
$$w_C = 0$$

Sample Problem

- Given $\theta_B = 0$, determine the relationship between m and P.



- Solution:



$$\theta_B = \frac{Pa^2}{2EI} - \frac{m \cdot 2a}{EI} = 0$$

$$\Rightarrow m = \frac{Pa}{4}$$

Deflection and Slope by Superposition

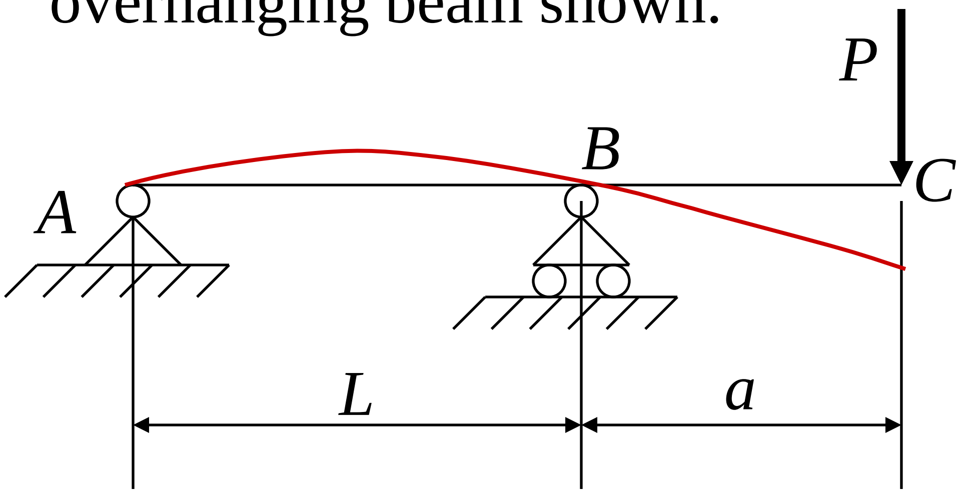
- **Superposition of Rigidized Structures:**

- Applicable to multi-span beams

- The total deflection of a multi-span beam under a given loading condition can be determined by superposing several beams corresponding to rigidizing all but one span of the beam, under the exactly same loading condition as the original beam.

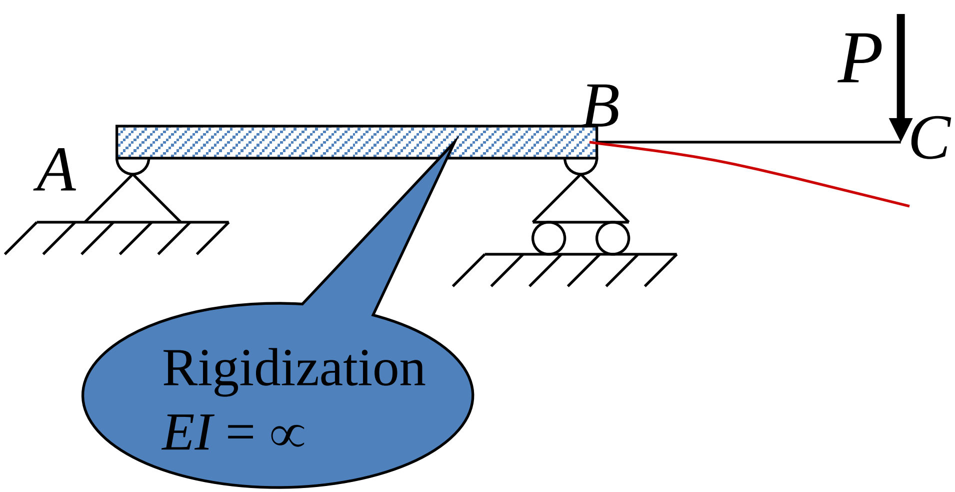
Sample Problem

- Find the deflection at section C of the simply supported overhanging beam shown.



- Solution

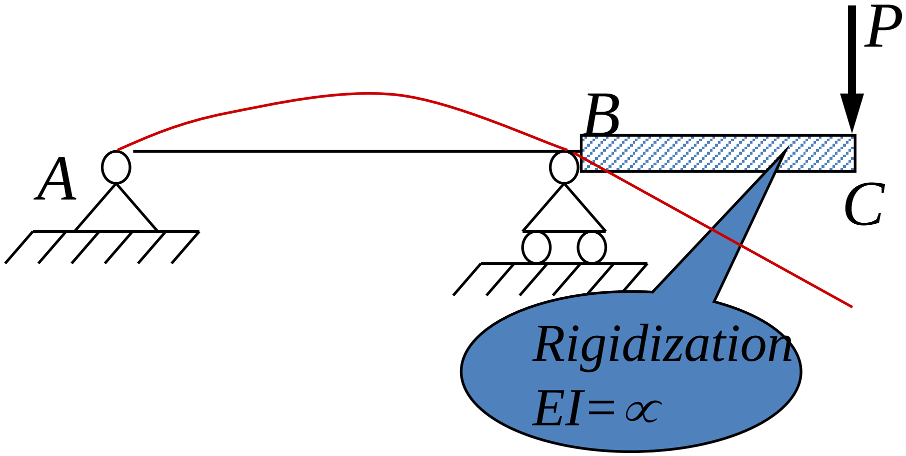
- Deflection at C due to rigidization of portion AB

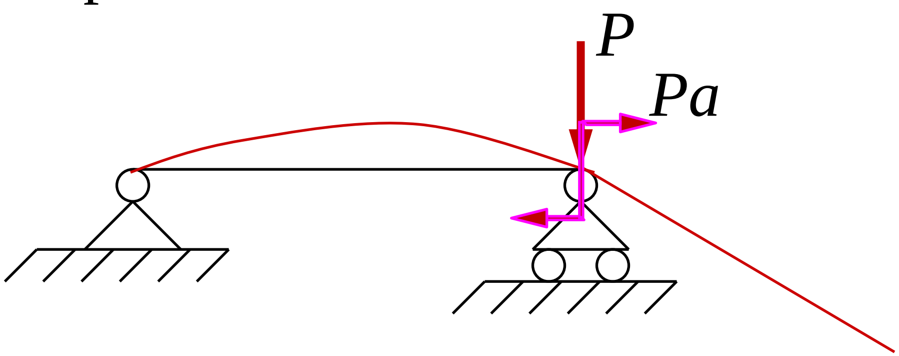


$$w_{c1} = \frac{Pa^3}{3EI}$$

$$\theta_{c1} = \frac{Pa^2}{2EI}$$

- Deflection at *C* due to rigidization of portion *BC*





$$w_{c2} = \theta_{B2} \cdot a = \frac{PaL}{3EI} a \qquad \theta_{C2} = \theta_{B2} = \frac{paL}{3EI}$$

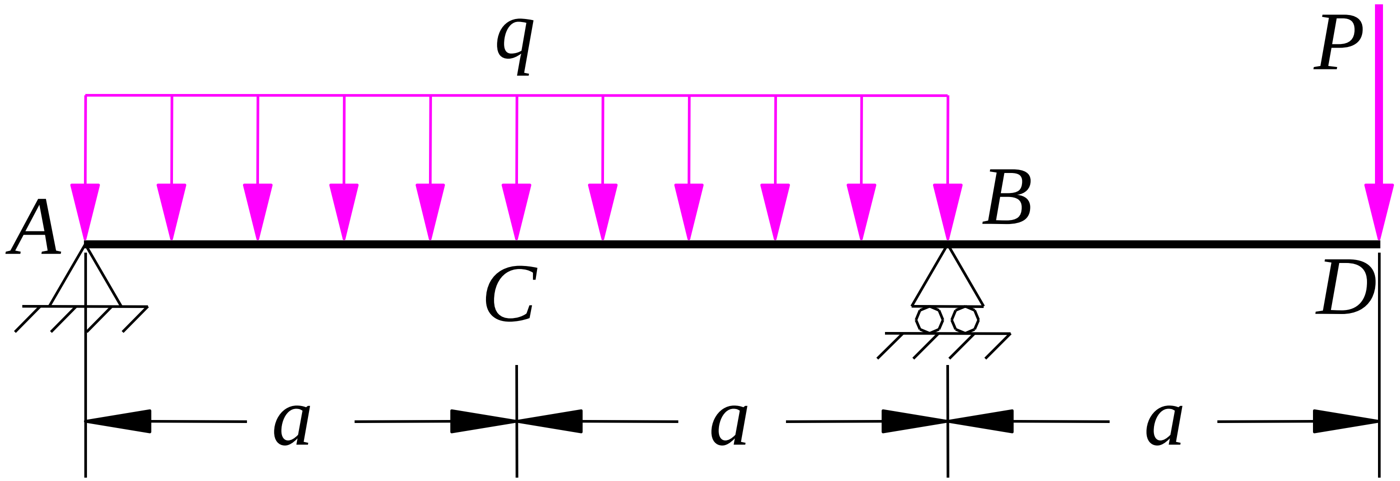
- Total deflection and slope at C:

$$w_c = w_{c1} + w_{c2}$$
$$= \frac{Pa^3}{3EI} + \frac{PaL}{3EI} a = \frac{Pa^2 (a+L)}{3EI}$$

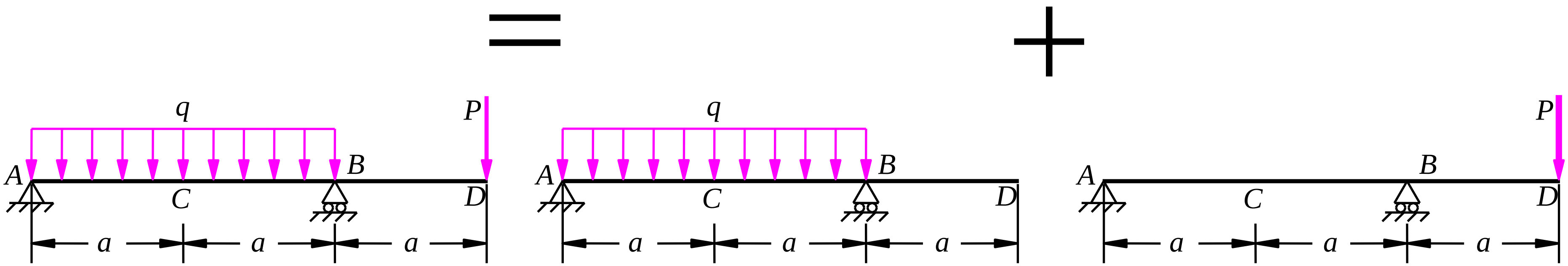
$$\theta_c = \theta_{c1} + \theta_{c2}$$
$$= \frac{Pa^2}{2EI} + \frac{PaL}{3EI} = \frac{Pa}{EI}\left(\frac{a}{2} + \frac{L}{3}\right)$$

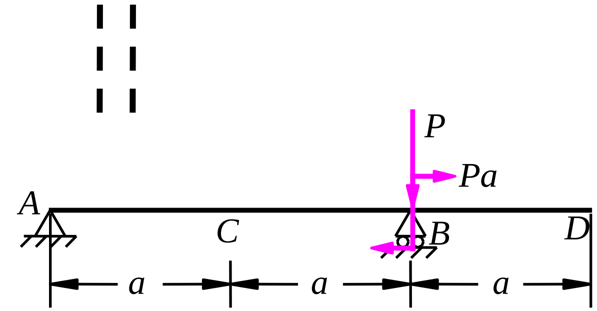
Superposition of Loads & Rigidized Structures

- Given $w_C = 0$, determine the relationship between P and q.



- Solution:



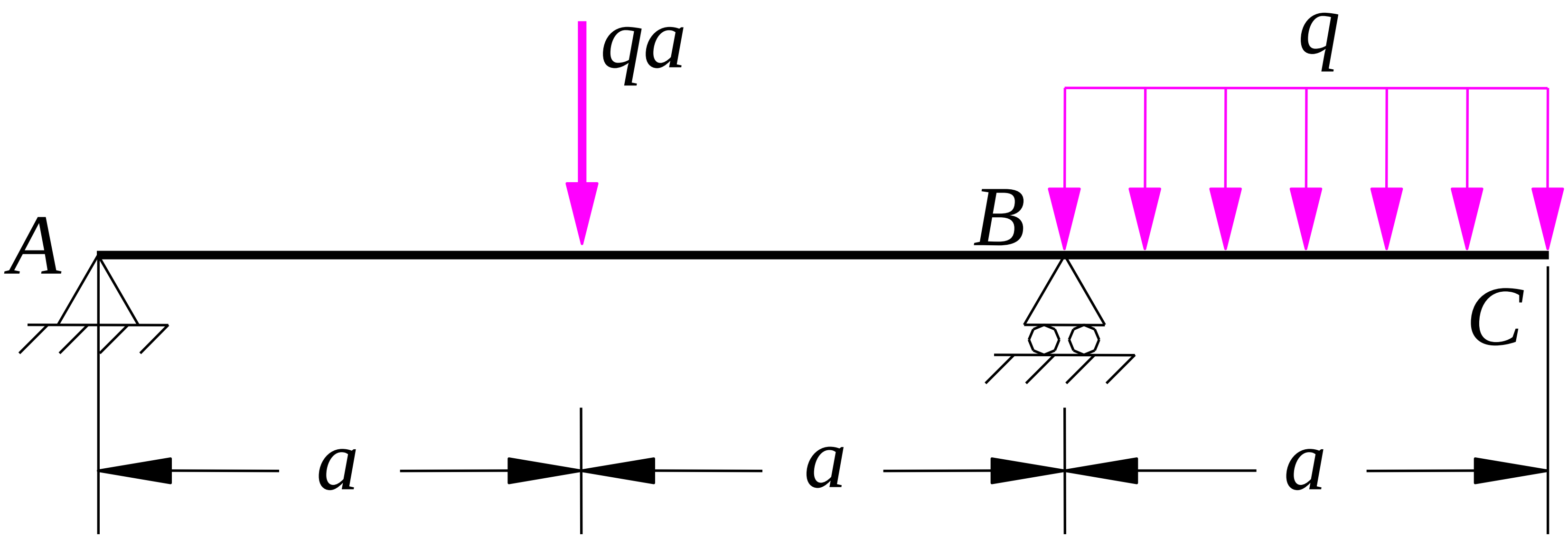


$$w_C = \frac{5q(2a)^4}{384EI} - \frac{Pa(2a)^2}{16EI} = 0$$

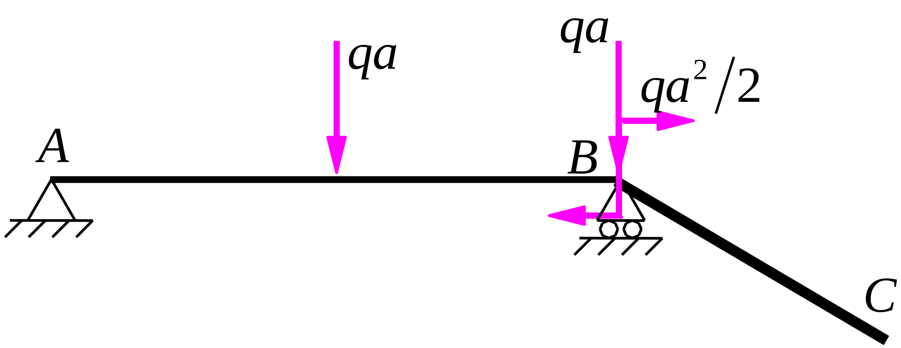
$$\Rightarrow P = \frac{5}{6}qa$$

Sample Problem

- Using the method of superposition find the deflection and slope at section *C* of the beam shown.



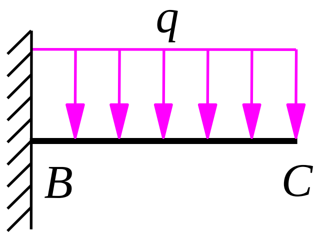
- Solution:



- Rigidizing *BC*

$$\theta_C = \theta_B = \frac{\frac{qa^2}{2} 2a}{3EI} - \frac{qa \cdot (2a)^2}{16EI} = \frac{qa^3}{12EI}$$

$$w_C = \theta_B \cdot a = \frac{qa^4}{12EI}$$



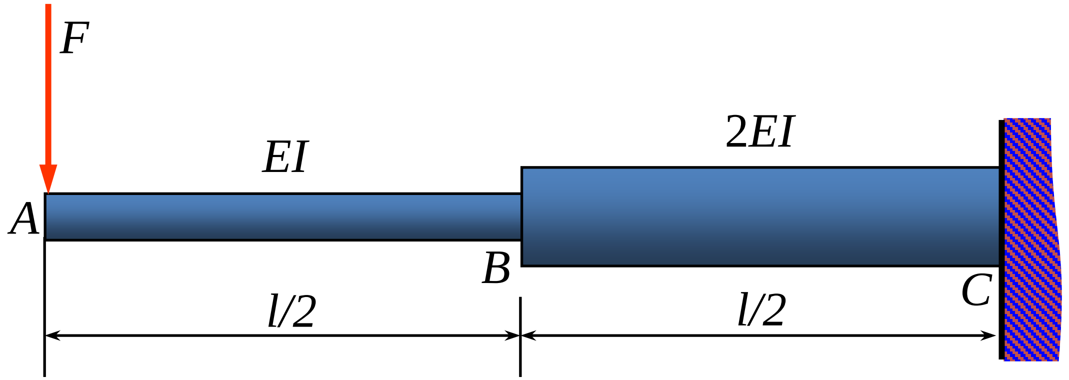
- Rigidizing *AB*

$$\theta_C = \frac{qa^3}{6EI}, w_C = \frac{qa^4}{8EI}$$

- Total: $\theta_C = \frac{qa^3}{12EI} + \frac{qa^3}{6EI} = \frac{qa^3}{4EI}, \quad w_C = \frac{qa^4}{12EI} + \frac{qa^4}{8EI} = \frac{5qa^4}{24EI}$

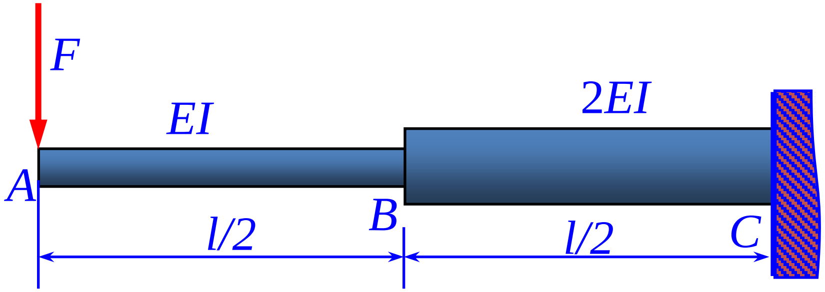
Sample Problem

- A stepped cantilever, as shown, is subjected to a concentrated load *F* at its free end. Find the deflection at the free end.

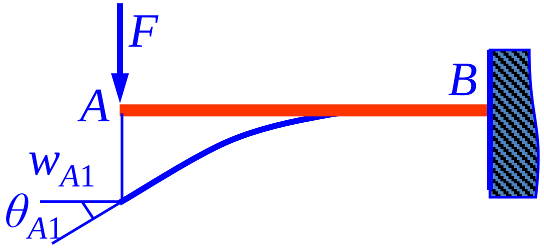


- Solution

- Rigidizing section *BC* makes *AB* a cantilever subjected to a concentrated load at its free end.



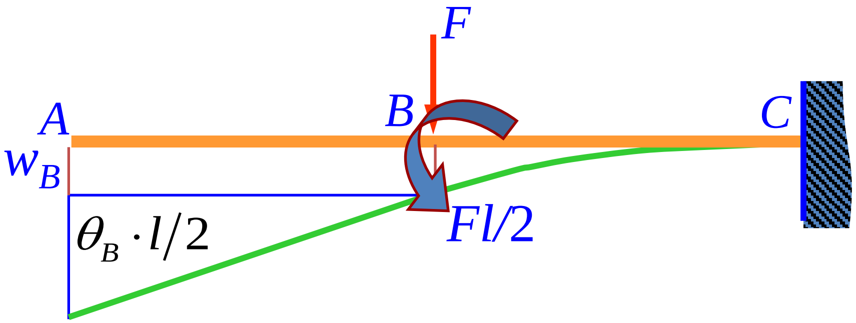
- Rigidizing section *AB* makes the whole beam a cantilever.



$$w_A = w_{A1} + w_{A2}$$

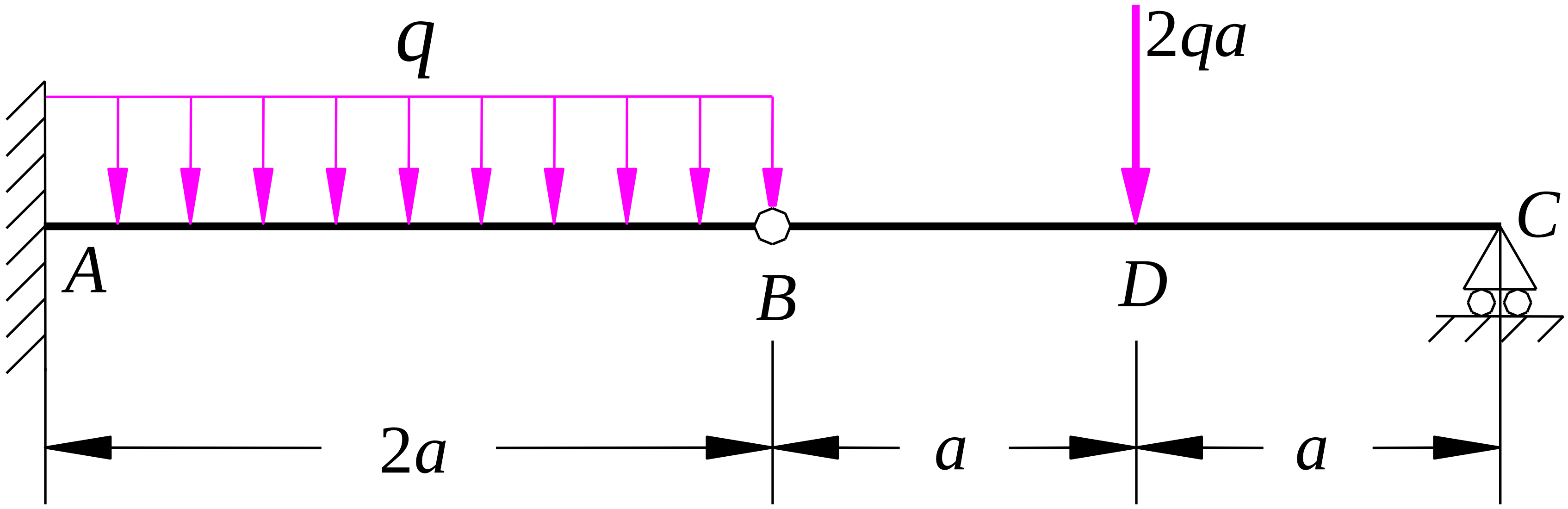
$$= w_{A1} + w_B + \theta_B \cdot \frac{l}{2}$$

$$= \frac{3Pl^3}{16EI}$$

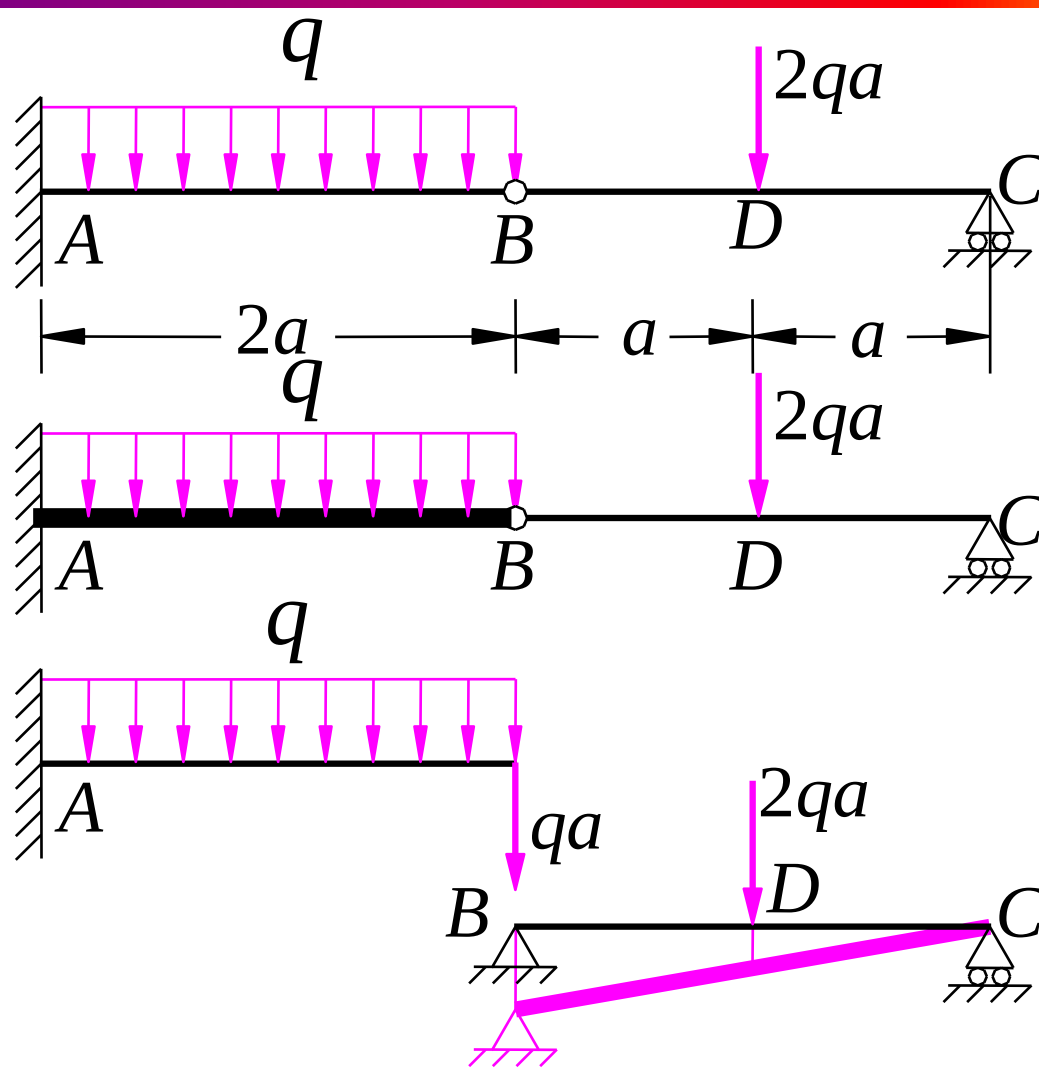


Sample Problem

- Find the deflections at sections *B* and *D* of the beam shown below.



- Solution



- Rigidizing *AB*

$$w_B = 0, \quad w_D = \frac{2qa(2a)^3}{48EI}$$

- Rigidizing *BC*

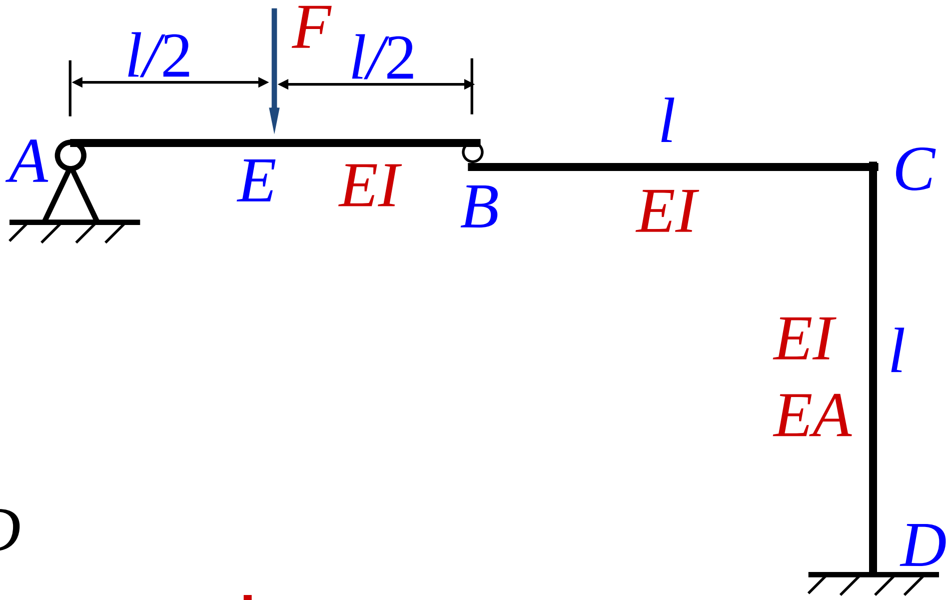
$$w_B = \frac{q(2a)^4}{8EI} + \frac{qa(2a)^3}{3EI} = \frac{14qa^4}{3EI}$$

$$w_D = \frac{w_B}{2} = \frac{7qa^4}{3EI}$$

- Total: $w_B = \dfrac{14qa^4}{3EI}, \quad w_D = \dfrac{7qa^4}{3EI} + \dfrac{2qa(2a)^3}{48EI} = \dfrac{8qa^4}{3EI}$

Sample Problem

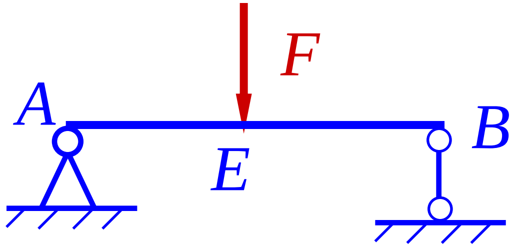
- For the structure composed of a beam and a frame shown, find the deflection at the center of the beam *AB*.
- Solution



- The deflection at section *E* is associated with the following deformations:
- Bending of beam *AB* itself.
- Bending of *BC*
- Compression and bending of *CD*

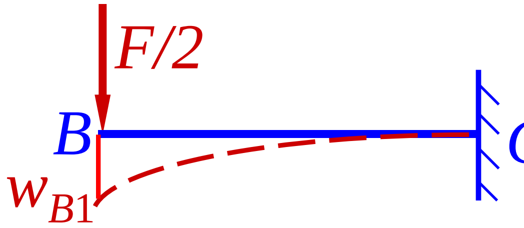
- Rigidize the frame (*BC*+*CD*)

$$w_{E1} = \frac{Fl^3}{48EI}$$



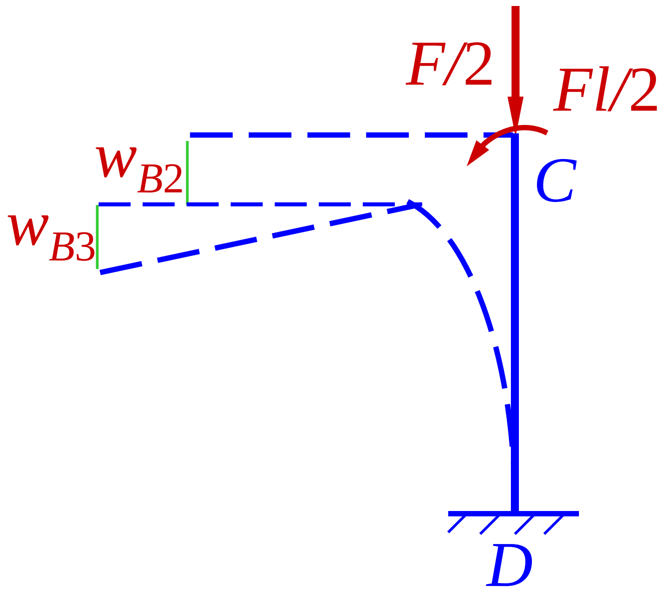
- Rigidize *AB* + *CD*

$$w_{E2} = \frac{1}{2} w_{B1} = \frac{1}{2} \frac{(\frac{F}{2})l^3}{3EI} = \frac{Fl^3}{12EI}$$



- Rigidize *AB* + *BC*

$$w_{E3} = \frac{1}{2}(w_{B2} + w_{B3})$$



$$w_{B2} = \frac{Fl}{2EA}$$

(Deflection at *B* due to the compression of *CD*)

$$w_{B3} = \theta_C l = \left[{(\tfrac{F}{2} l)l} \middle/ {EI} \right] l = \frac{Fl^3}{2EI}$$

(Deflection at *B* due to the bending of *CD*)

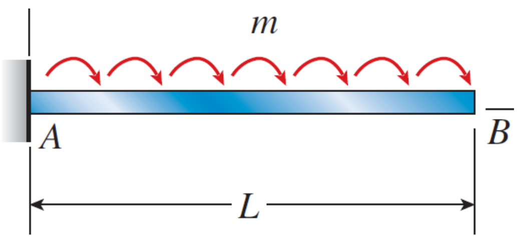
$$\Rightarrow w_{E3} = \frac{1}{2}\left(\frac{Fl}{2EA} + \frac{Fl^3}{2EI}\right)$$

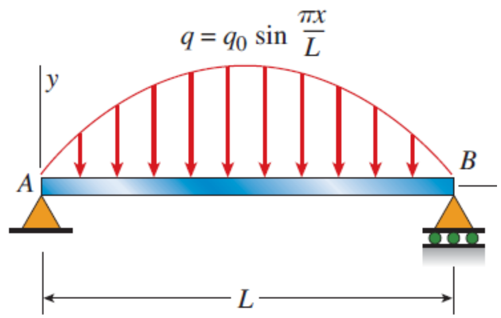
- Deflection at section *E* via superposition:

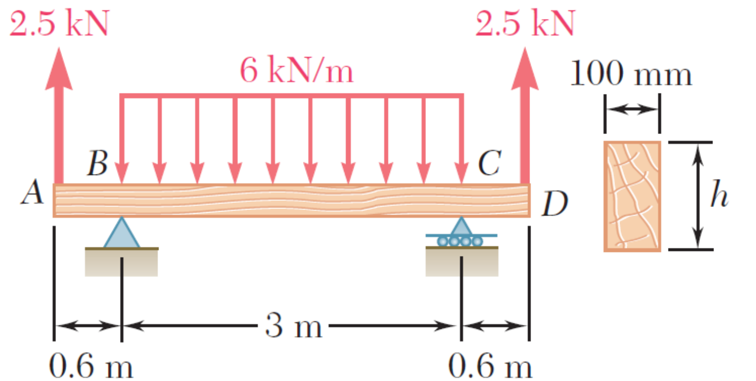
$$w_E = w_{E1} + w_{E2} + w_{E3} = \frac{Fl^3}{EI}\left(\frac{1}{48} + \frac{1}{12} + \frac{1}{4}\right) + \frac{Fl}{4EA}$$

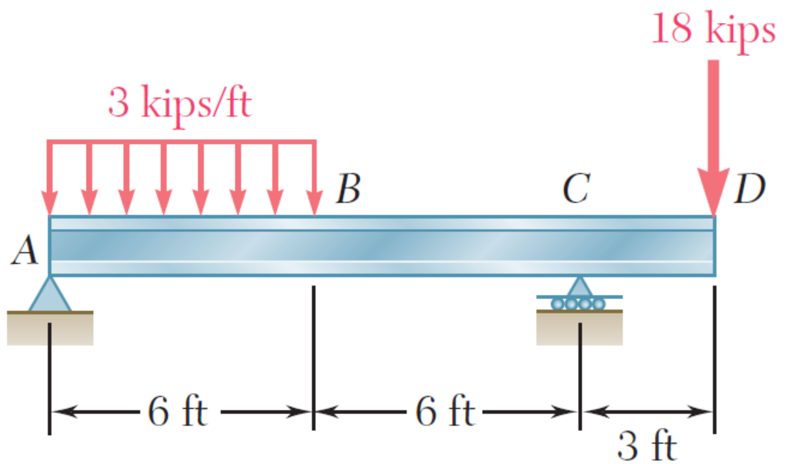
$$= \frac{17Fl^3}{48EI} + \frac{Fl}{4EA}$$

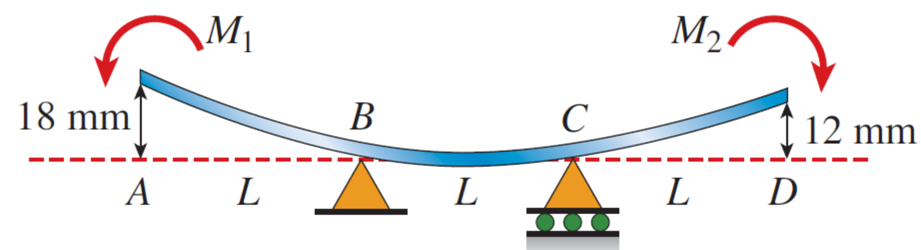
More Examples

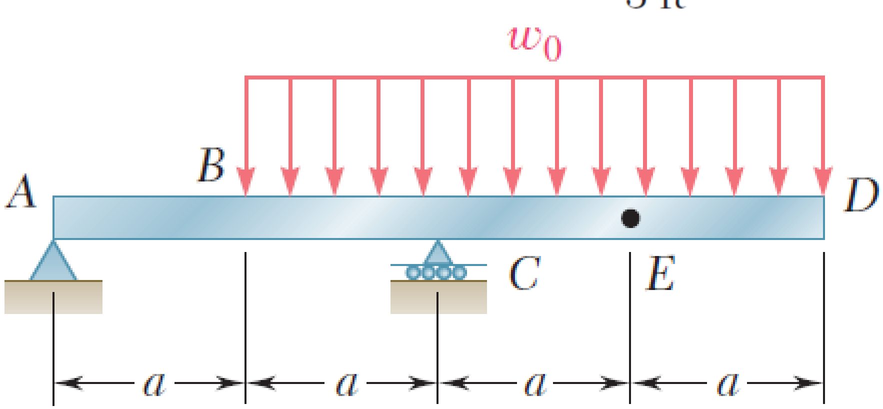


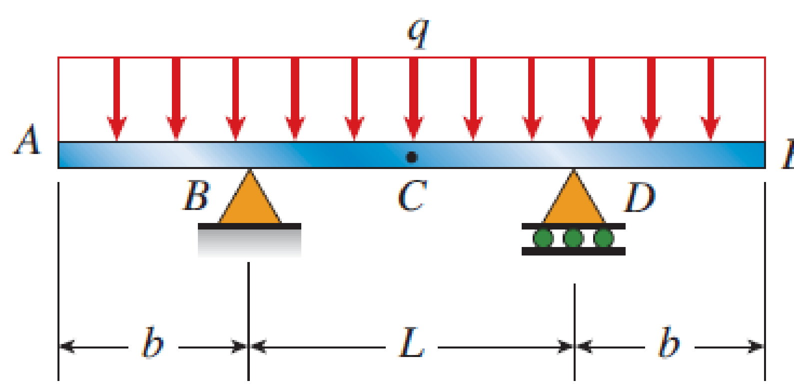




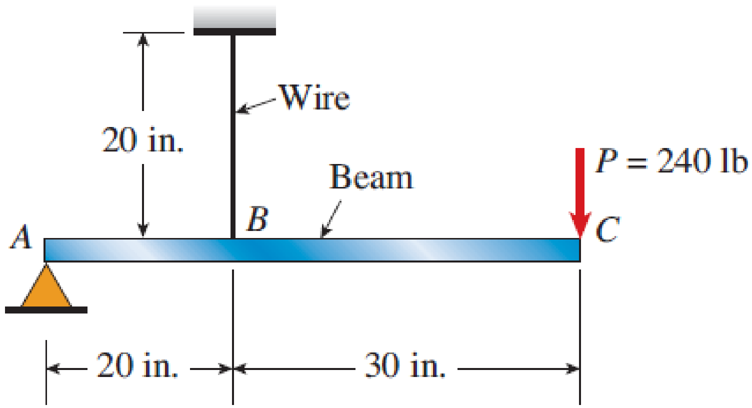


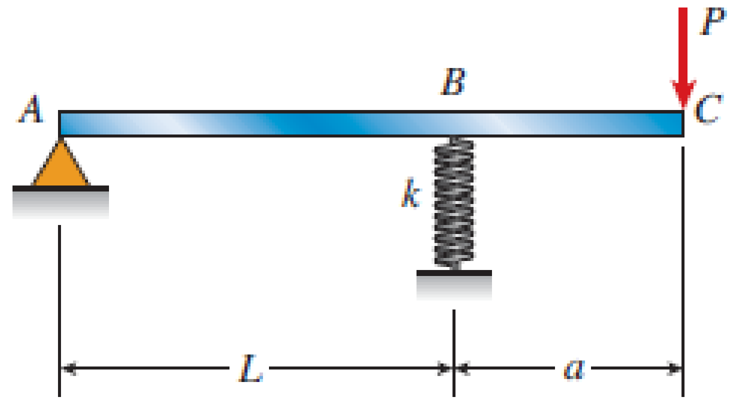


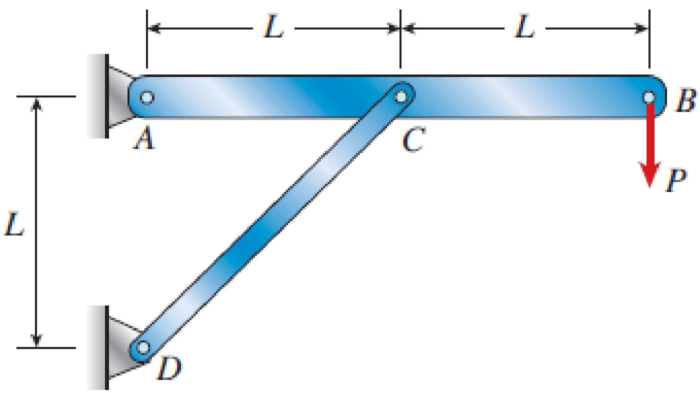


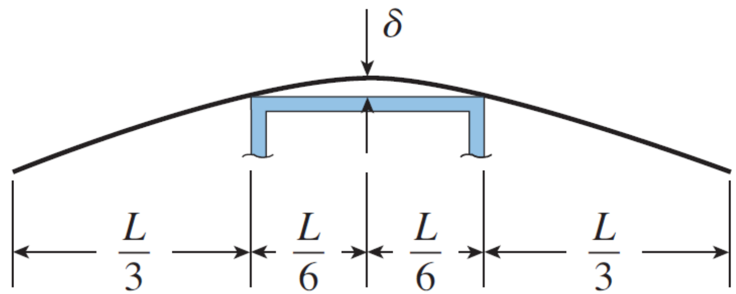


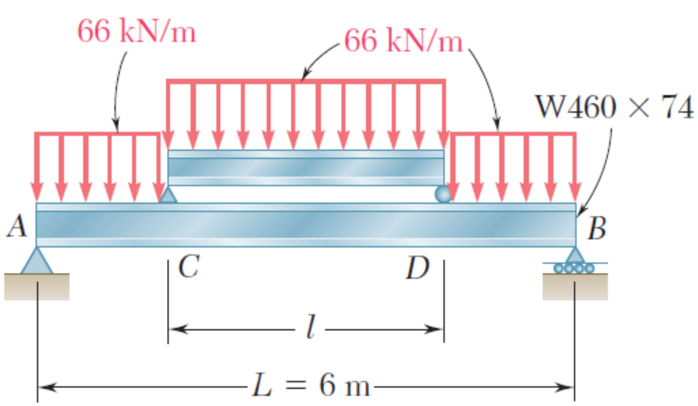
More Examples

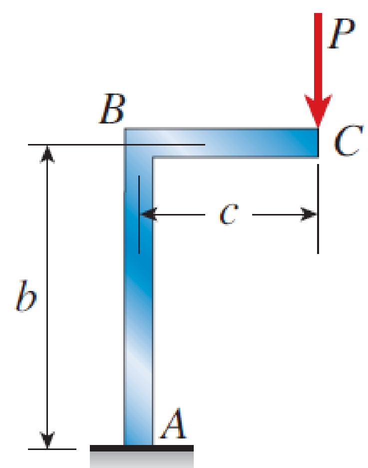




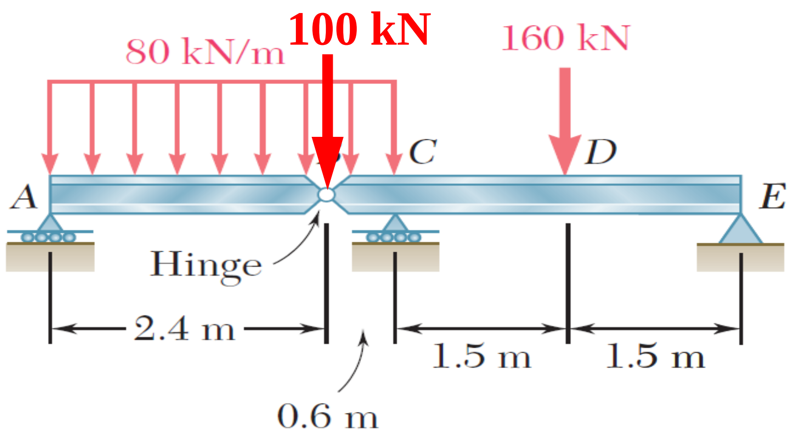


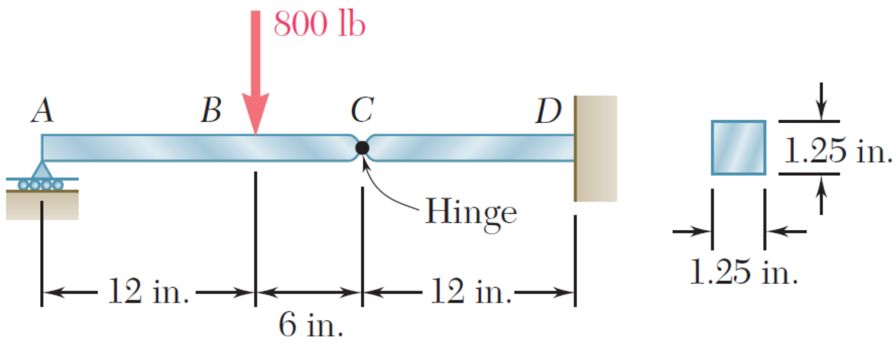


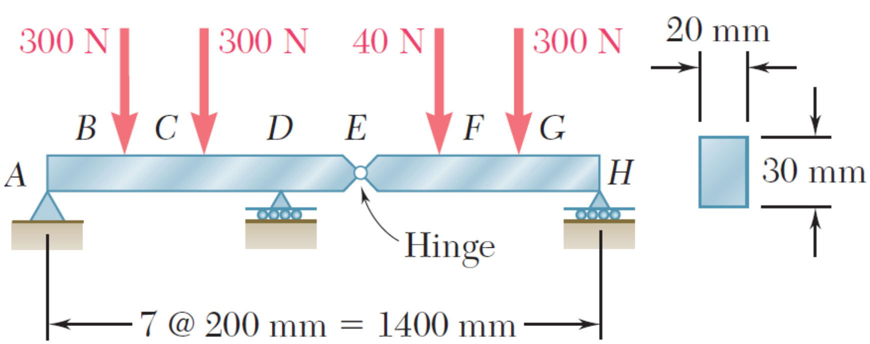


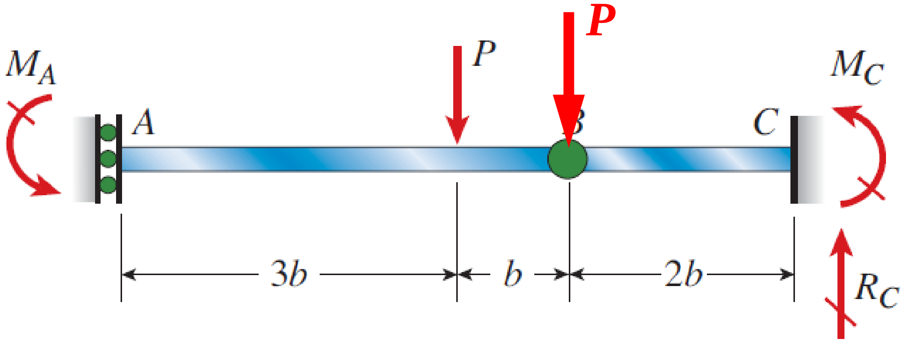


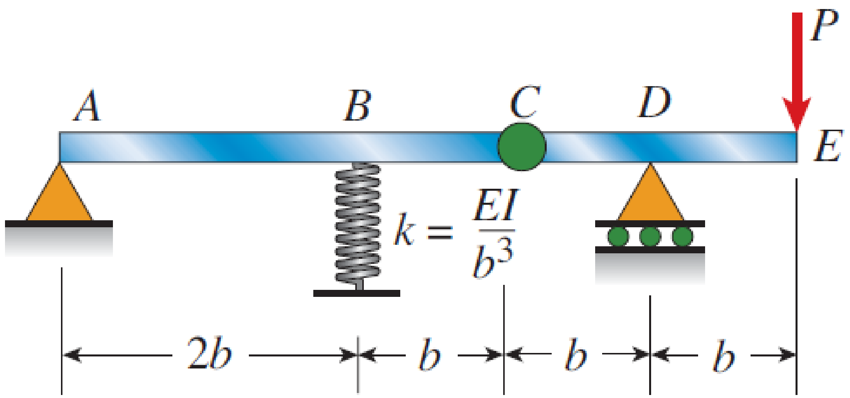
More Examples

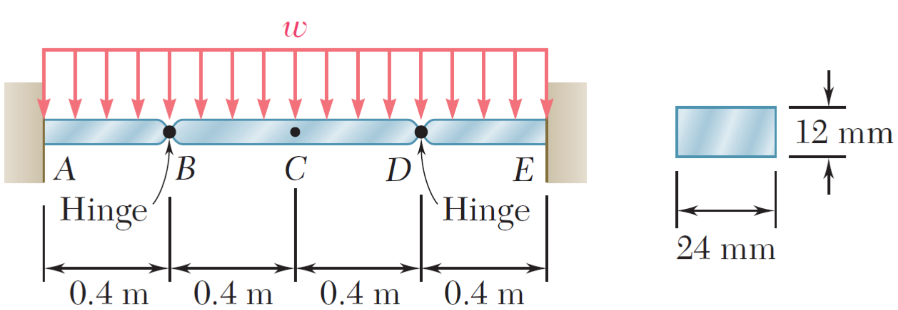






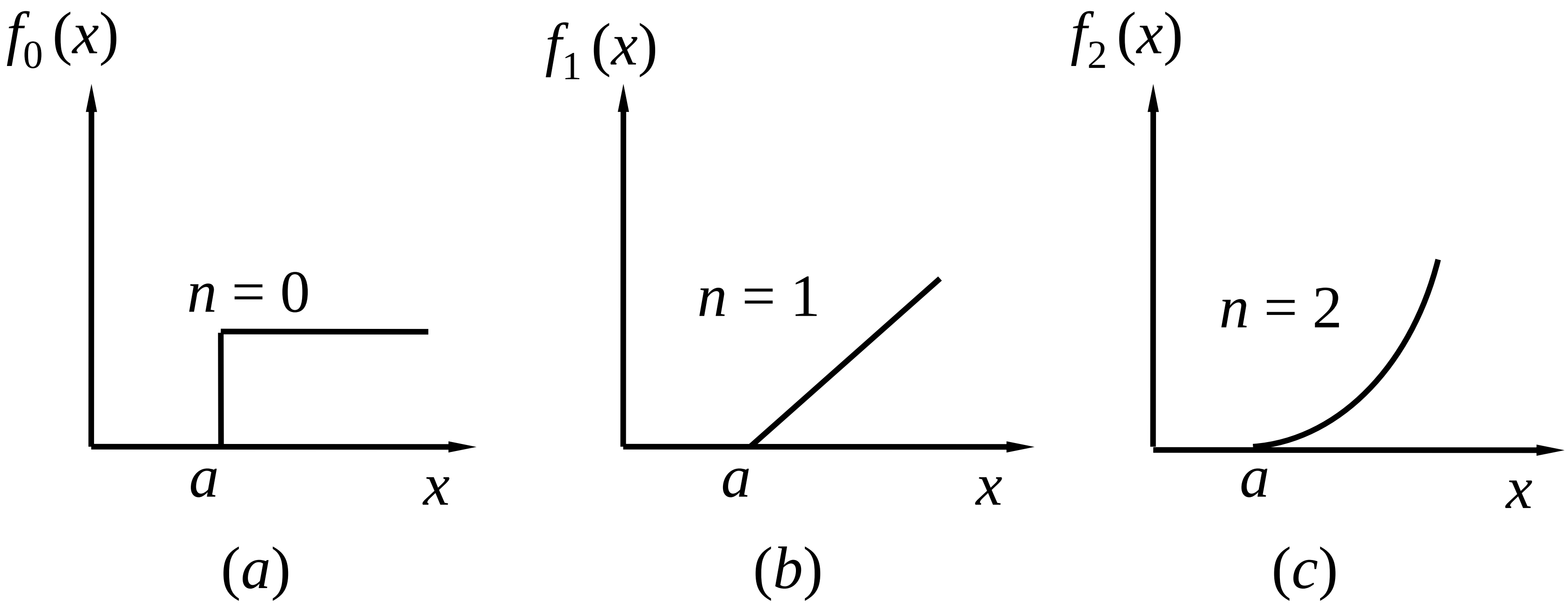






Singular / Discontinuity Functions

$$f_n(x) = \langle x-a \rangle^n = \begin{cases} (x-a)^n & x \geq a \\ 0 & x < a \end{cases}$$



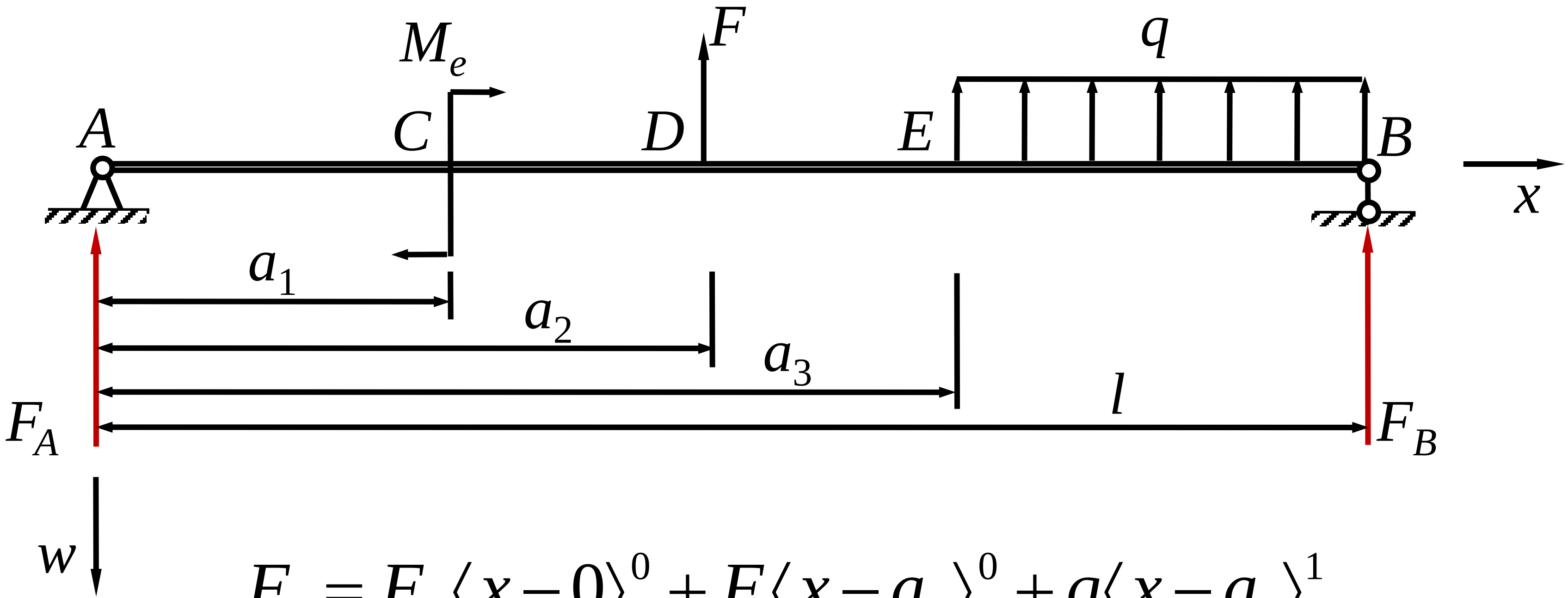
Calculus of Singular Functions

$$f_n(x) = \langle x-a \rangle^n = \begin{cases} (x-a)^n & x \geq a \\ 0 & x < a \end{cases}$$

$$\int \langle x-a \rangle^n dx = \frac{1}{n+1} \langle x-a \rangle^{n+1} + C \qquad n \geq 0$$

$$\frac{d}{dx} \langle x-a \rangle^n = \begin{cases} 0 & n = 0 \\ n \langle x-a \rangle^{n-1} & n \geq 1 \end{cases}$$

Equations of Shearing Forces & Bending Moments



$$F_S = F_A\langle x-0\rangle^0 + F\langle x-a_2\rangle^0 + q\langle x-a_3\rangle^1$$
$$= F_A + F\langle x-a_2\rangle^0 + q\langle x-a_3\rangle^1$$

$$M = F_A\langle x-0\rangle^1 + M_e\langle x-a_1\rangle^0 + F\langle x-a_2\rangle^1 + \frac{q}{2}\langle x-a_3\rangle^2$$
$$= F_A x + M_e\langle x-a_1\rangle^0 + F\langle x-a_2\rangle^1 + \frac{q}{2}\langle x-a_3\rangle^2$$

Boundary Conditions

- Denote the shearing force and bending moment at the left boundary as F_{S0} and M_0

- Generalized equation of shearing forces

$$F_s(x) = F_{S0} + F\langle x - a_2\rangle^0 + q\langle x - a_3\rangle^1$$

- Generalized equation of bending moment

$$M(x) = M_0 + F_{S0}x + M_e\langle x - a_1\rangle^0 + F\langle x - a_2\rangle^1 + \frac{q}{2}\langle x - a_3\rangle^2$$

Deflection and Slope by Singular Functions

$$EIw'' = -M(x)$$

$$M(x) = M_0 + F_{S0}x + M_e\langle x - a_1\rangle^0 + F\langle x - a_2\rangle^1 + \frac{q}{2}\langle x - a_3\rangle^2$$

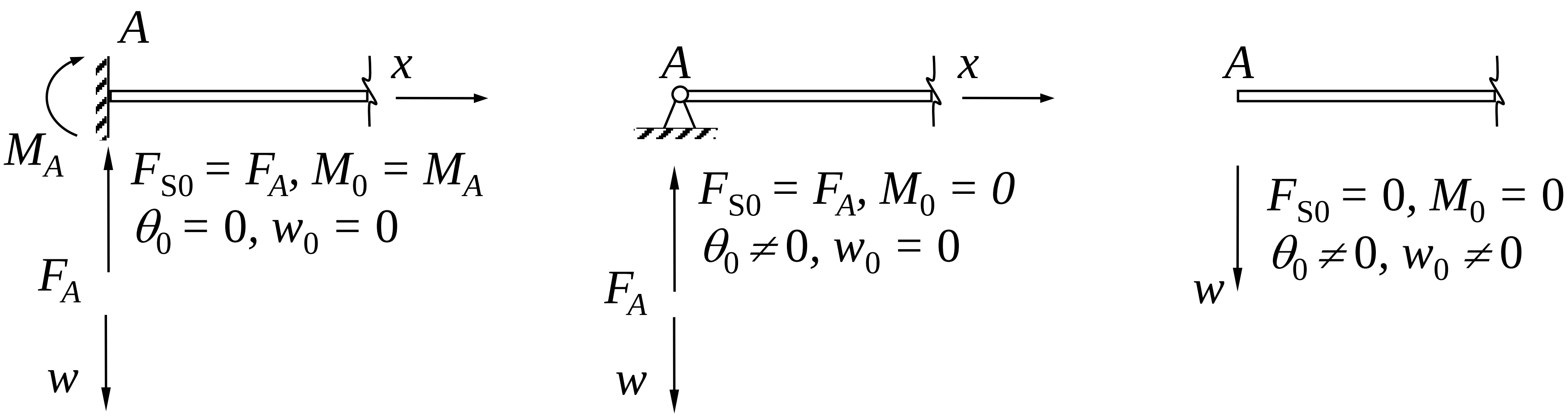
$$EI\theta = -M_0x - \frac{F_{S0}}{2}x^2 - M_e\langle x - a_1\rangle^1 - \frac{F}{2}\langle x - a_2\rangle^2 - \frac{q}{6}\langle x - a_3\rangle^3 + C_1$$

$$EIw = -\frac{M_0}{2}x^2 - \frac{F_{S0}}{6}x^3 - \frac{M_e}{2}\langle x - a_1\rangle^2 - \frac{F}{6}\langle x - a_2\rangle^3 - \frac{q}{24}\langle x - a_3\rangle^4 + C_1x + C_2$$

$$C_1 = EI\theta_0, \qquad C_2 = EIw_0$$

Boundary Values

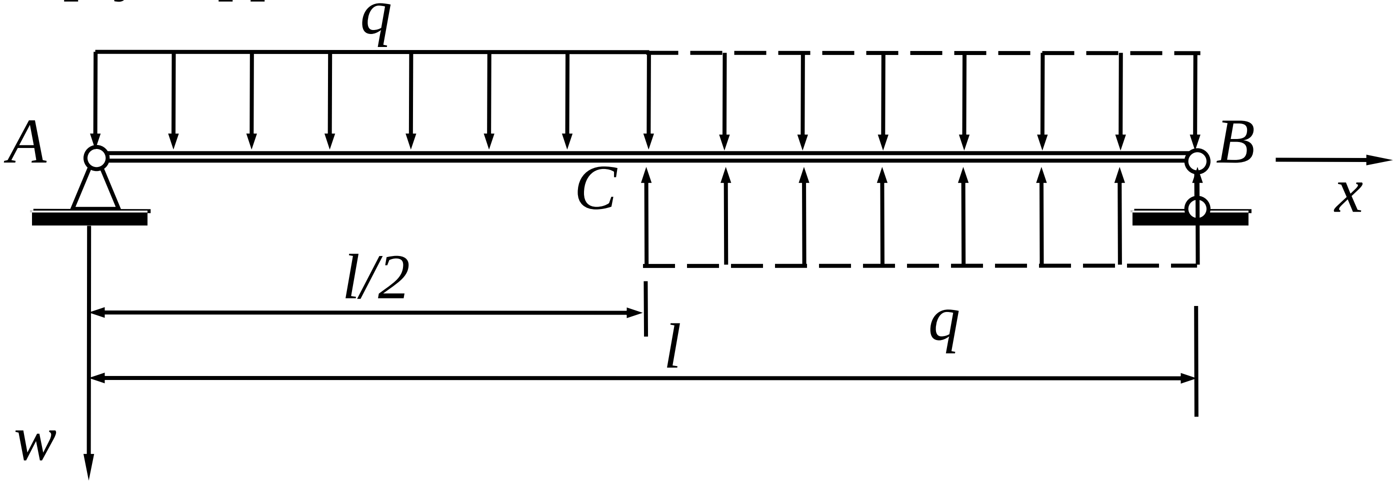
- F_{S0}, M_0, θ_0 and w_0 denote the boundary values of shearing force, bending moment, deflection and slope



(a) Fixed support (b) Hinged support (c) Free end

Sample Problem

- Find the deflection at section *C* and the slopes at sections *A* and *B* for the simply supported beam shown.



- Solution

1. Equations of deflection and slope

$$EI\theta = EI\theta_0 - M_0 x - \frac{F_{S0}}{2!}x^2 + \frac{q}{3!}x^3 - \frac{q}{3!}\langle x - \frac{l}{2}\rangle^3$$

$$EIw = EIw_0 + EI\theta_0 x - \frac{M_0}{2!}x^2 - \frac{F_{S0}}{3!}x^3 + \frac{q}{4!}x^4 - \frac{q}{4!}\langle x - \frac{l}{2}\rangle^4$$

- Determine boundary values

$$F_{S0} = F_A = \frac{3}{8}ql \ , \ M_0 = 0 \ , \ w_0 = 0$$

- Determine θ_0 from the boundary condition at the movable hinged support *B*:

$$0 = EIw\big|_{x=l} = EI\theta_0 l - \frac{3ql}{8}\cdot\frac{l^3}{6} + \frac{q}{24}l^4 - \frac{q}{24}\frac{l^4}{16} = 0 \Rightarrow \theta_0 = \frac{3ql^3}{128EI}$$

$$\Rightarrow \begin{cases} EI\theta = \dfrac{3ql^3}{128} - \dfrac{3ql}{8}\cdot\dfrac{x^2}{2} + \dfrac{q}{6}x^3 - \dfrac{q}{6}\left\langle x - \dfrac{l}{2}\right\rangle^3 \\ EIw = \dfrac{3ql^3 x}{128} - \dfrac{3ql}{8}\cdot\dfrac{x^3}{6} + \dfrac{q}{24}x^4 - \dfrac{q}{24}\left\langle x - \dfrac{l}{2}\right\rangle^4 \end{cases}$$

2. The Slopes θ_A and θ_B and the deflection w_C

$$EI\theta = \frac{3ql^3}{128} - \frac{3ql}{8}\cdot\frac{x^2}{2} + \frac{q}{6}x^3 - \frac{q}{6}\langle x - \frac{l}{2}\rangle^3$$

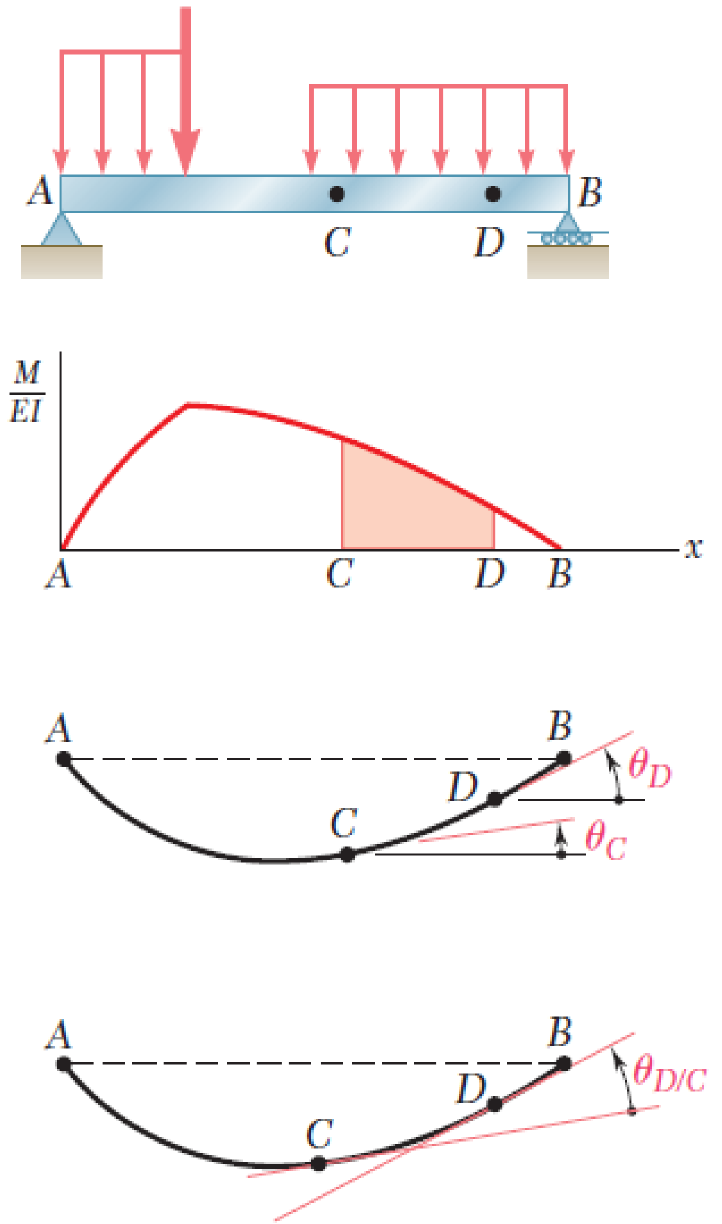
$$EIw = \frac{3ql^3 x}{128} - \frac{3ql}{8}\cdot\frac{x^3}{6} + \frac{q}{24}x^4 - \frac{q}{24}\langle x - \frac{l}{2}\rangle^4$$

$$\theta_A = \theta_0 = \frac{3ql^3}{128EI} \curvearrowright$$

$$\theta_B = \theta\big|_{x=l} = \frac{ql^3}{EI}\left(\frac{3}{128} - \frac{3}{16} + \frac{1}{6} - \frac{1}{6\cdot 8}\right) = -\frac{7ql^3}{384EI} \curvearrowleft$$

$$w_C = w\big|_{x=\frac{l}{2}} = \frac{ql^4}{EI}\left(\frac{3}{128\cdot 2} - \frac{3}{48\cdot 8} + \frac{1}{24\cdot 16}\right) = \frac{5ql^4}{768EI} \downarrow$$

Moment-Area Theorems



- Geometric properties of the elastic curve can be used to determine deflection and slope.

- Consider a beam subjected to arbitrary loading,

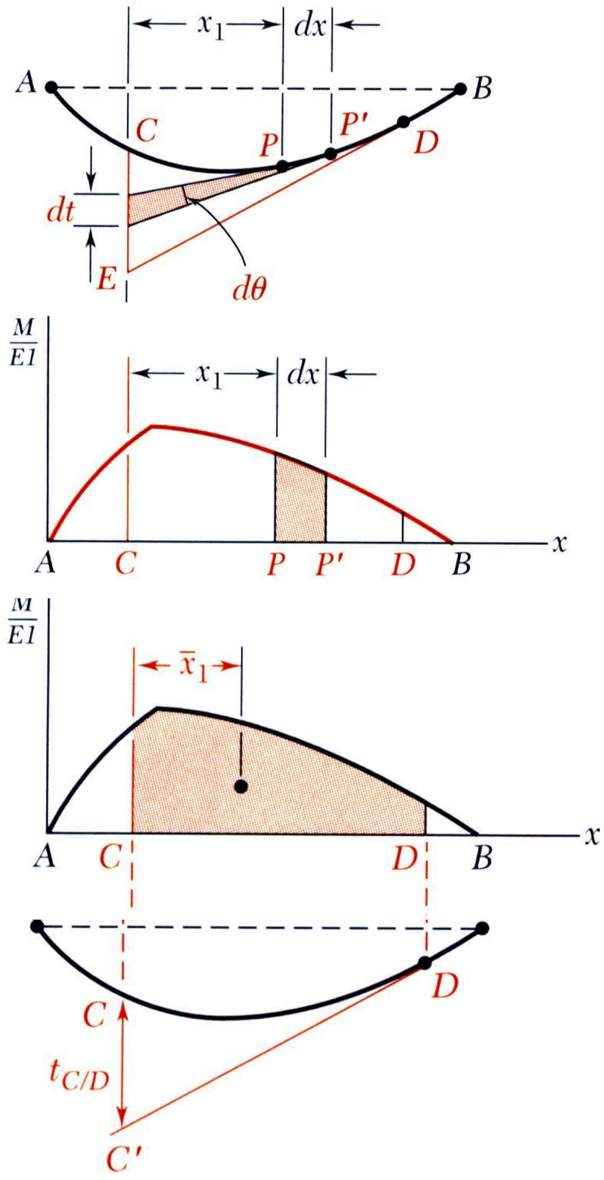
$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI} \Rightarrow \int_{\theta_C}^{\theta_D} d\theta = \int_{x_C}^{x_D} \frac{M}{EI} dx$$

$$\Rightarrow \theta_D - \theta_C = \int_{x_C}^{x_D} \frac{M}{EI} dx$$

- *First Moment-Area Theorem*:

$\theta_{D/C} =$ area under (*M/EI*) diagram between *C* and *D*.

Moment-Area Theorems



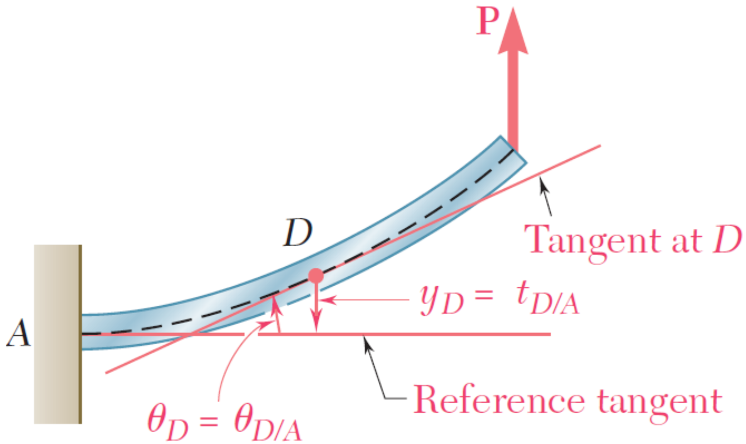
- Tangents to the elastic curve at *P* and *P'* intercept a segment of length d*t* on the vertical through *C*.

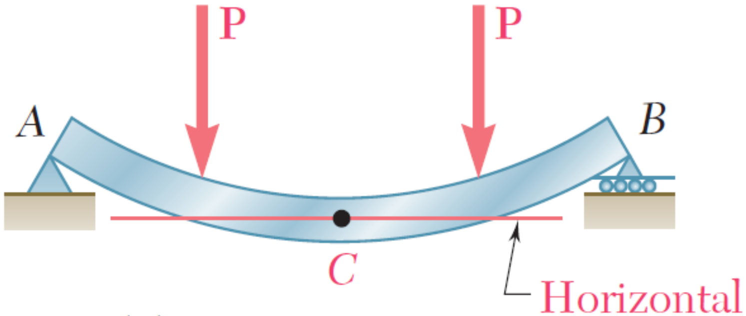
$$dt = x_1 d\theta = x_1 \frac{M}{EI} dx$$

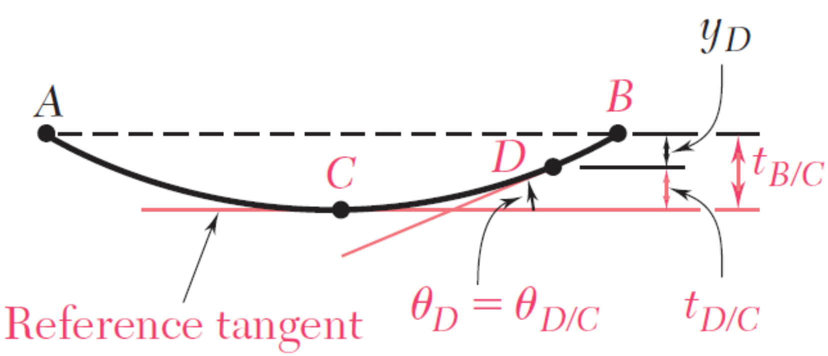
$$t_{C/D} = \int_{x_C}^{x_D} x_1 \frac{M}{EI} dx = \text{tangential deviation of } C \text{ with respect to } D$$

- *Second Moment-Area Theorem*:
  The tangential deviation of *C* with respect to *D* is equal to the first moment with respect to a vertical axis through *C* of the area under the (*M/EI*) diagram between *C* and *D*.

Application to Cantilevers & Beams under Symmetric Loading







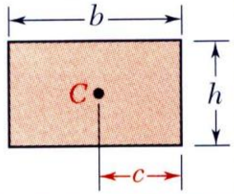
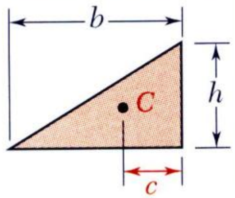
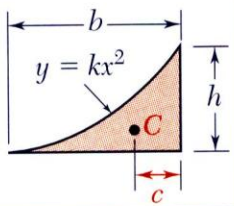
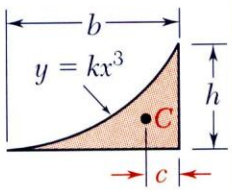
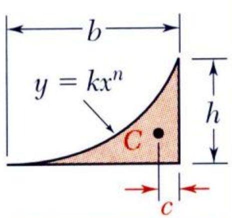
- Cantilever beam - Select tangent at A as the reference.

$$\theta_A = 0, \quad y_A = 0$$
$$\theta_D = \theta_{D/A}$$
$$y_D = t_{D/A}$$

- Simply supported, symmetrically loaded beam - select tangent at C as the reference.

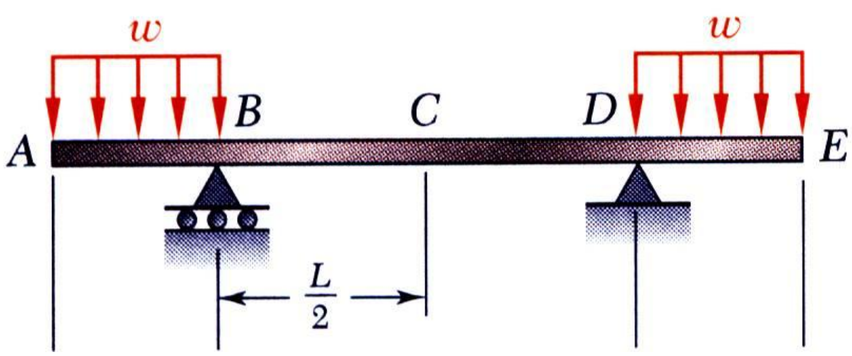
$$\theta_C = 0, \quad y_C = y_{\max}$$
$$\theta_D = \theta_{D/C}$$
$$y_B - y_C = -y_C = t_{B/C}$$
$$y_D - y_C = t_{D/C}$$

Bending Moment Diagrams by Parts

Shape		Area	c
Rectangle		bh	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$
Parabolic spandrel		$\frac{bh}{3}$	$\frac{b}{4}$
Cubic spandrel		$\frac{bh}{4}$	$\frac{b}{5}$
General spandrel		$\frac{bh}{n+1}$	$\frac{b}{n+2}$

- Determination of the change of slope and the tangential deviation is simplified if the effect of each load is evaluated separately.
- Construct a separate (*M/EI*) diagram for each load.
  - The change of slope, $\theta_{D/C}$, is obtained by adding the areas under the diagrams.
  - The tangential deviation, $t_{D/C}$ is obtained by adding the first moments of the areas with respect to a vertical axis through *D*.
- Bending moment diagram constructed from individual loads is said to be *drawn by parts*.

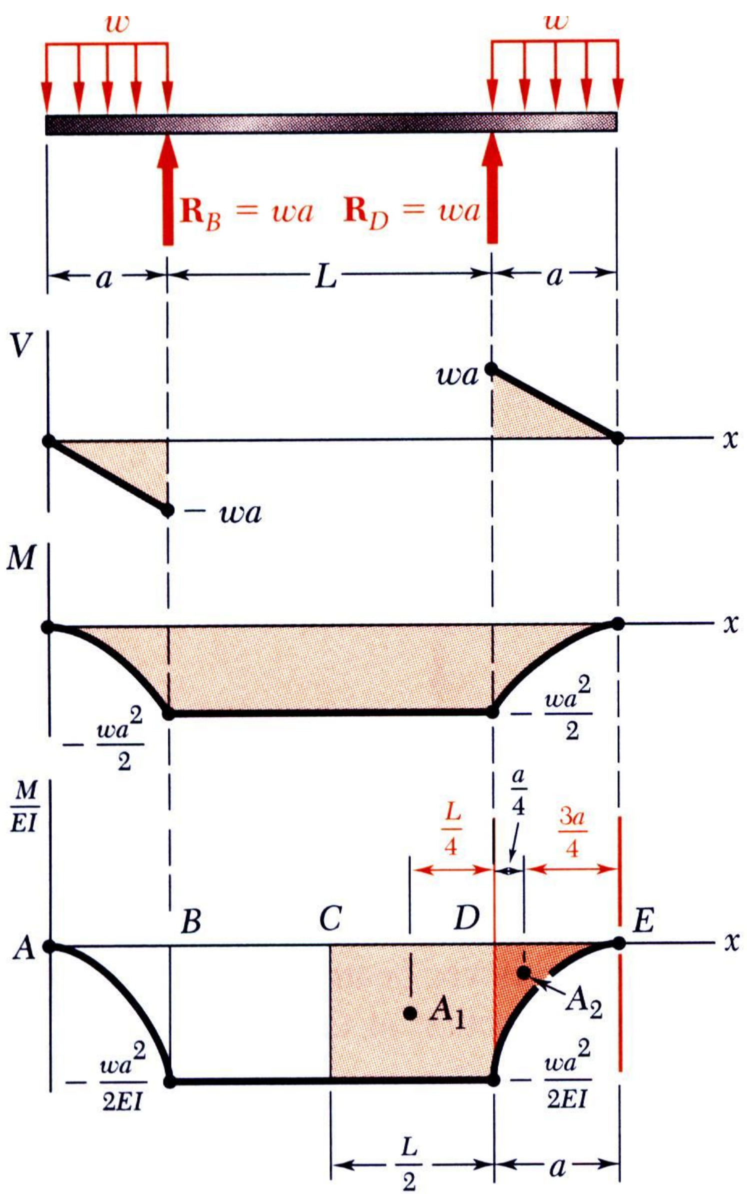
Sample Problem



For the prismatic beam shown, determine the deflection and slope at *E*.

SOLUTION:

- Determine the reactions at supports.
- Construct shear, bending moment and (*M/EI*) diagrams.
- Taking the tangent at *C* as the reference, evaluate the slope and tangential deviations at *E*.



SOLUTION:

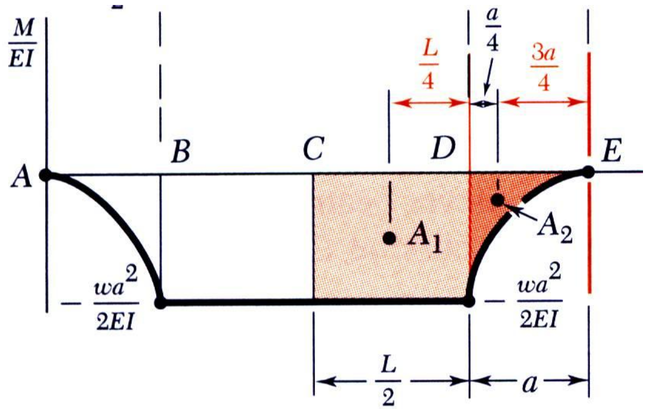
- Determine the reactions at supports.

$$R_B = R_D = wa$$

- Construct shear, bending moment and (*M/EI*) diagrams.

$$A_1 = -\frac{wa^2}{2EI}\left(\frac{L}{2}\right) = -\frac{wa^2 L}{4EI}$$

$$A_2 = -\frac{1}{3}\left(\frac{wa^2}{2EI}\right)(a) = -\frac{wa^3}{6EI}$$

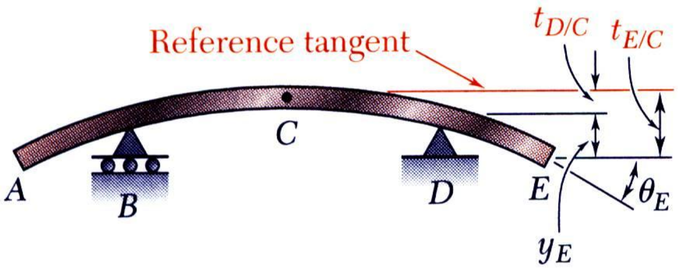


- Slope at E:

$$\theta_E = \theta_C + \theta_{E/C} = \theta_{E/C}$$

$$= A_1 + A_2 = -\frac{wa^2L}{4EI} - \frac{wa^3}{6EI}$$

$$\theta_E = -\frac{wa^2}{12EI}(3L + 2a)$$



- Deflection at E:

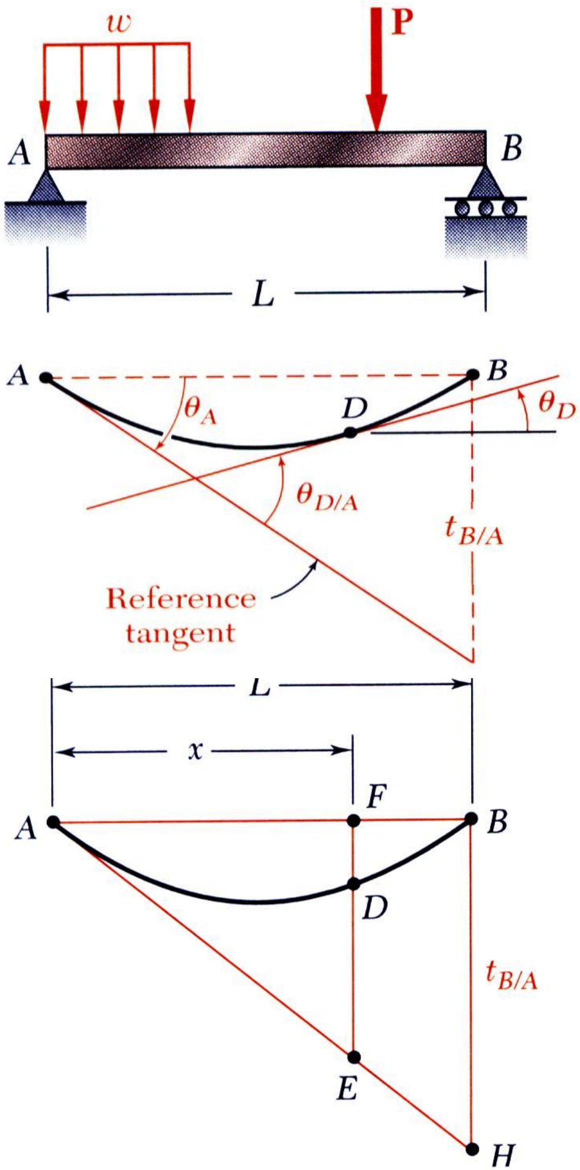
$$y_E = t_{E/D} = t_{E/C} - t_{D/C}$$

$$= \left[A_1\left(a + \frac{L}{4}\right) + A_2\left(\frac{3a}{4}\right)\right] - \left[A_1\left(\frac{L}{4}\right)\right]$$

$$= \left[-\frac{wa^3L}{4EI} - \frac{wa^2L^2}{16EI} - \frac{wa^4}{8EI}\right] - \left[-\frac{wa^2L^2}{16EI}\right]$$

$$y_E = -\frac{wa^3}{8EI}(2L + a)$$

Application to Beams under Unsymmetric Loadings



- Define reference tangent at support *A*. Evaluate θ_A by determining the tangential deviation at *B* with respect to *A*.

$$\theta_A = -\frac{t_{B/A}}{L}$$

- The slope at other points is found with respect to reference tangent.

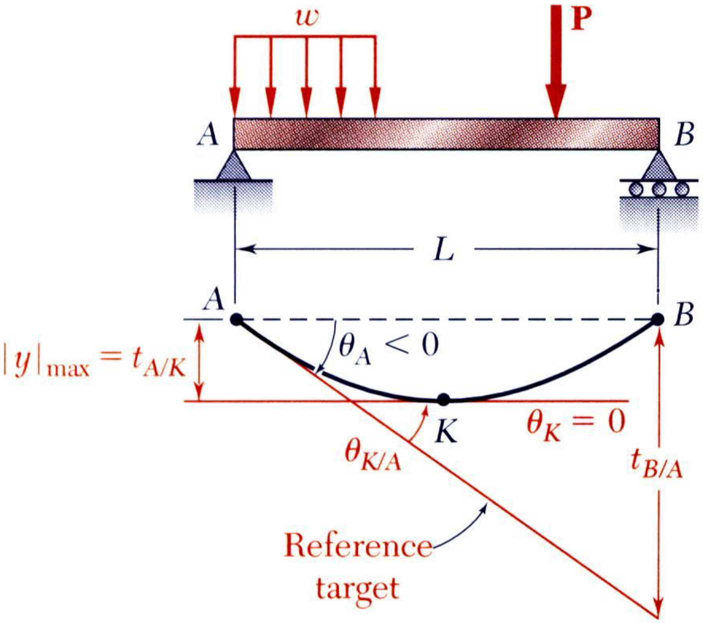
$$\theta_D = \theta_A + \theta_{D/A}$$

- The deflection at *D* is found from the tangential deviation at *D*.

$$\frac{FE}{t_{B/A}} = \frac{x}{L} \qquad FE = \frac{x}{L}\, t_{B/A}$$

$$y_D = -FD = -(FE - DE) = -\left(\frac{x}{L} t_{B/A} - t_{D/A}\right)$$

Maximum Deflection



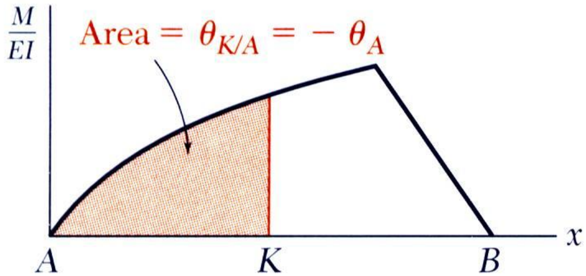
- Maximum deflection occurs at point *K* where the tangent is horizontal.

$$\theta_A = -\frac{t_{B/A}}{L}$$

$$\theta_K = 0 = \theta_A + \theta_{K/A}$$

$$\theta_{K/A} = -\theta_A$$

- Point *K* may be determined by measuring an area under the (*M/EI*) diagram equal to $-\theta_A$.



- Obtain w_{max} by computing the first moment with respect to the vertical axis through ***A*** of the area between *A* and *K*.

Stiffness Condition

- $w_{\max} \leq [w]$
- $\theta_{\max} \leq [\theta]$

$W_{\max}$: Maximum deflection

$\theta_{\max}$: Maximum slope

$[w]$, $[\theta]$: Maximum allowable deflection and slope

- Stiffness calculation include:
  - Stiffness check
  - Rational design of cross-sections
  - Find the maximum allowable loads

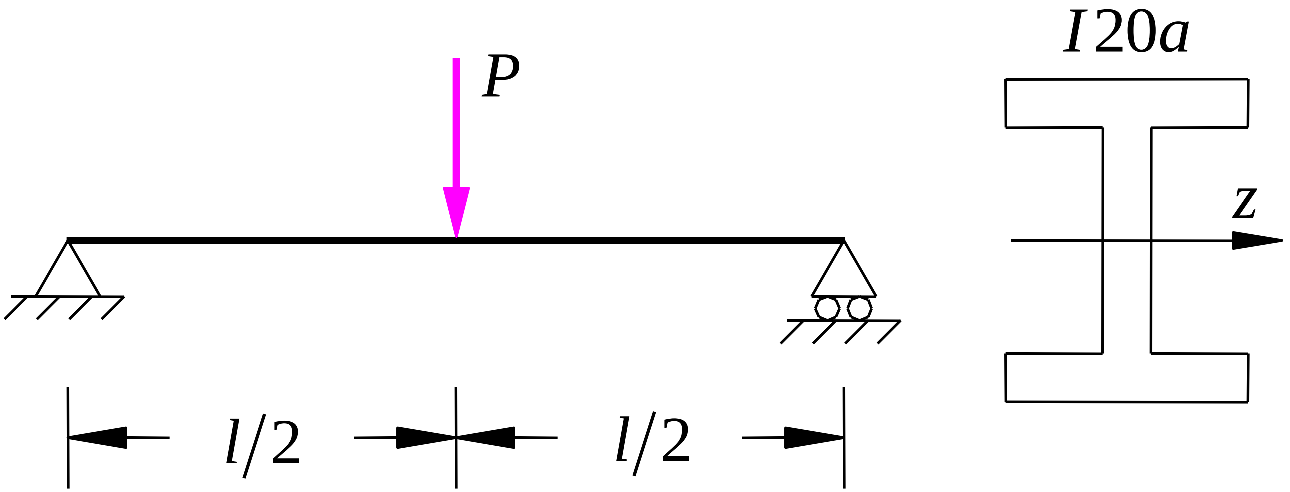
Ways to Increase Flexural Rigidity

- Deformation of beams under bending is influenced by not only beam supports and loading condition, but also beam material, cross-section size and shape, and beam span.

  - Increase EI
  - Decrease beam span / increase supports
  - Improve loading
  - Rational design of cross-sections

Sample Problem

- Given: l = 8 m, I_z = 2370 cm^4, W_z = 237 cm^3, [w] = l/500, E = 200 Gpa, [σ] = 100 Mpa.
- Find: 1. the maximum allowable load from the stiffness condition; 2. Strength check.



- Solution

$$w_{\max} = \frac{Pl^3}{48EI} \le [w] = \frac{l}{500}$$

$$\Rightarrow P \le \frac{48EI}{500l^2} = 7.11\text{kN}$$

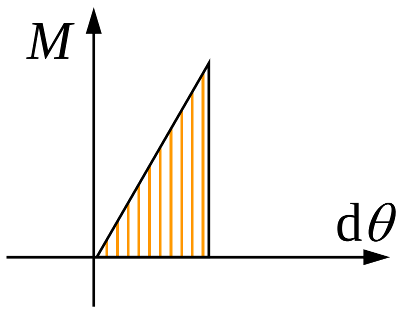
$$\Rightarrow [P] = 7.11\,\text{kN}$$

$$\sigma_{\max} = \frac{M_{\max}}{W_z} = \frac{Pl}{4W_z} = 60\ \text{MPa} \le [\sigma]$$

- The strength condition is satisfied.

Bending Strain Energy

- Strain energy density: $u = \frac{1}{2}\sigma\varepsilon = \frac{\sigma^2}{2E} = \frac{E\varepsilon^2}{2}$

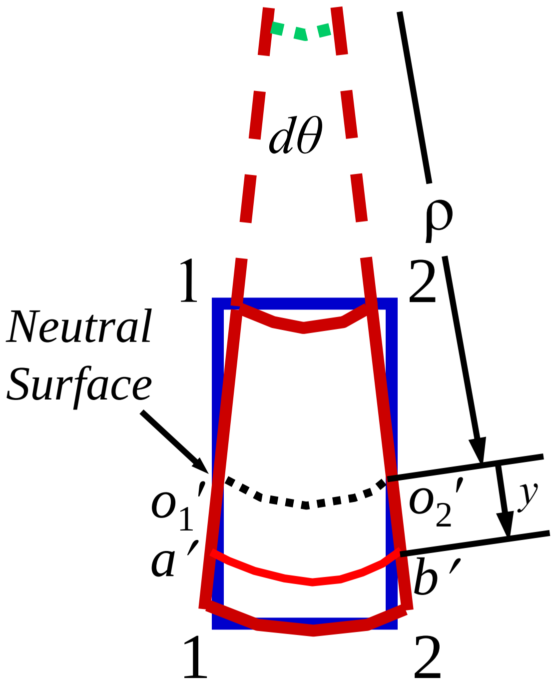


- Total strain energy calculated from density

$$U = \int \frac{\sigma^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

$$= \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right) dx = \int_0^L \frac{M^2}{2EI} dx$$

- Total strain energy calculated from work done by bending moment w.r.t. rotation



$$\frac{1}{\rho} = \frac{M}{EI}, \quad dx = \rho d\theta \quad \Rightarrow d\theta = \frac{Mdx}{EI}$$

$$dU = \frac{1}{2} M d\theta = \frac{M^2 dx}{2EI} \quad \Rightarrow U = \int_0^L \frac{M^2(x)}{2EI} dx$$

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- Deflection & Slope by Integration（积分法求挠度和转角）
- Boundary Conditions（边界条件）
- Symmetry Conditions（对称性条件）
- Continuity Conditions（连续性条件）
- Direct Integration from Distributed Loads（直接由分布荷载积分求挠度和转角）
- Direct Integration from Transverse Loads（直接由剪力积分求挠度和转角）
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- Deflection & Slope by Moment-Area Theorems（图乘法求挠度和转角）
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- Ways to Increase Flexural Rigidity（梁的刚度优化设计）
- Bending Strain Energy（弯曲应变能）