Stress State

mi@seu.edu.cn

Contents

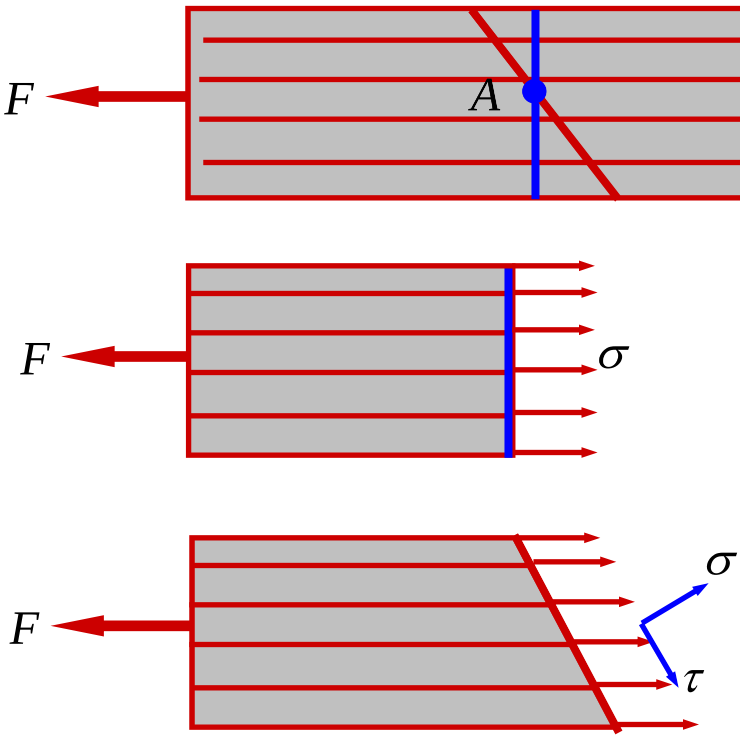
Contents

- 3-D Cauchy's Relation（三维柯西关系）
- Stress Transformation of General 3-D Stresses（空间应力转换）
- Application of Mohr's Circle in 3-D（莫尔圆在三维应力状态中的应用）
- Generalized Hooke's Law（广义胡克定律）
- The Relation among *E*, *v* and *G*（杨氏模量、泊松系数及剪切模量间的关系）
- Fiber-reinforced Composites（纤维增强复合材料）
- Strain Energy Density under Generalized Stress States（一般应力状态下的应变能）
- Volume Strain, Mean Stress and Bulk Modulus（体应变、平均应力和体积模量）
- Volumetric & Distortion Energy Density（体积应变能密度和畸变能密度）

The Stress State of a Point

- The stress state at a point
  - is defined as the collection of the stress distributions on all planes passing through the point.
  - can be analyzed via the stress distributions acting on a differential cube, arbitrarily oriented with reference coordinates
- Method of differential cubes with:
  - infinitesimal side length
  - uniformly distributed stress on each of the six surfaces
  - equal and opposite stress on parallel sides

Stresses on Oblique Planes

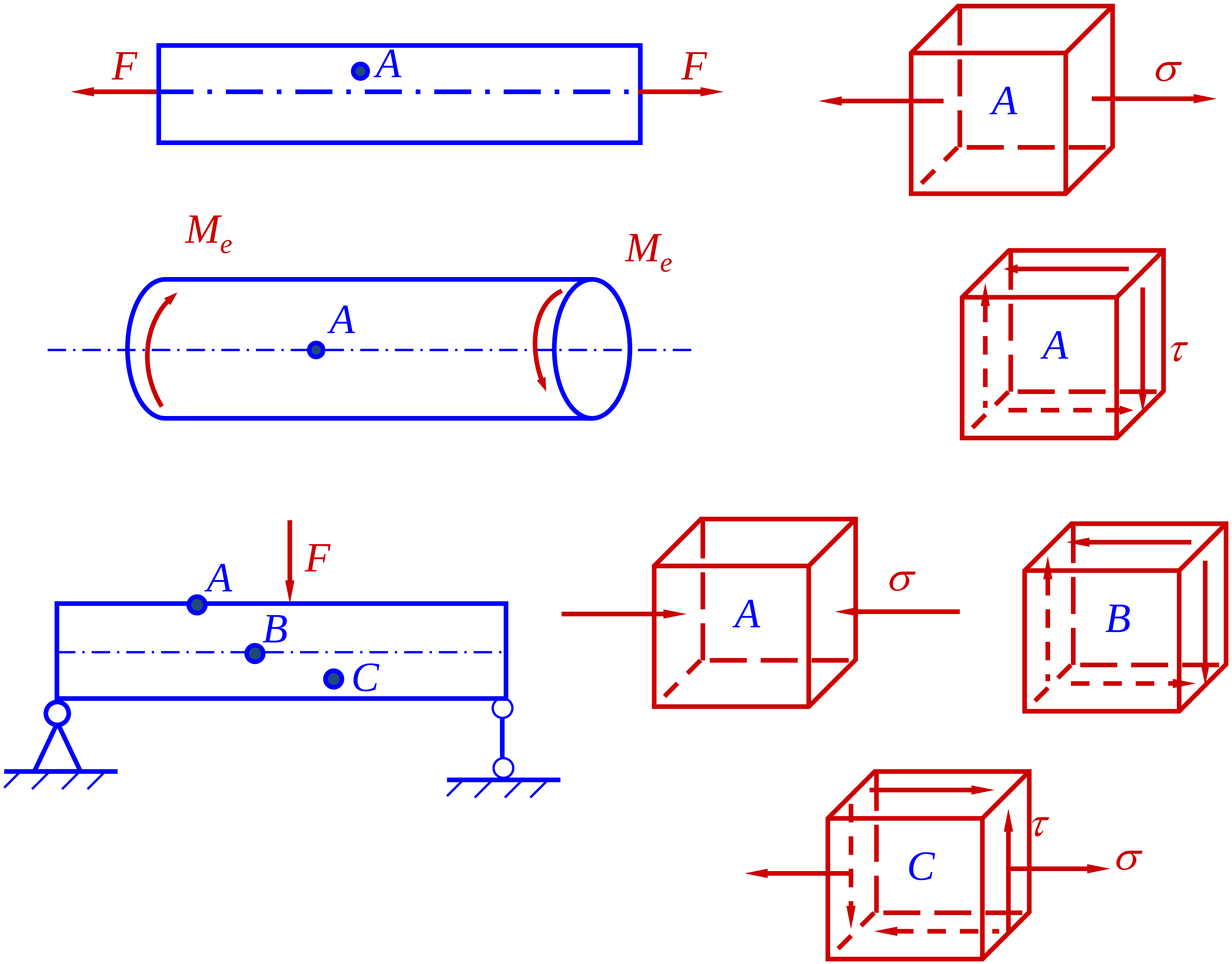


- The stress distribution at a point depends on the plane orientation along which the point is examined.

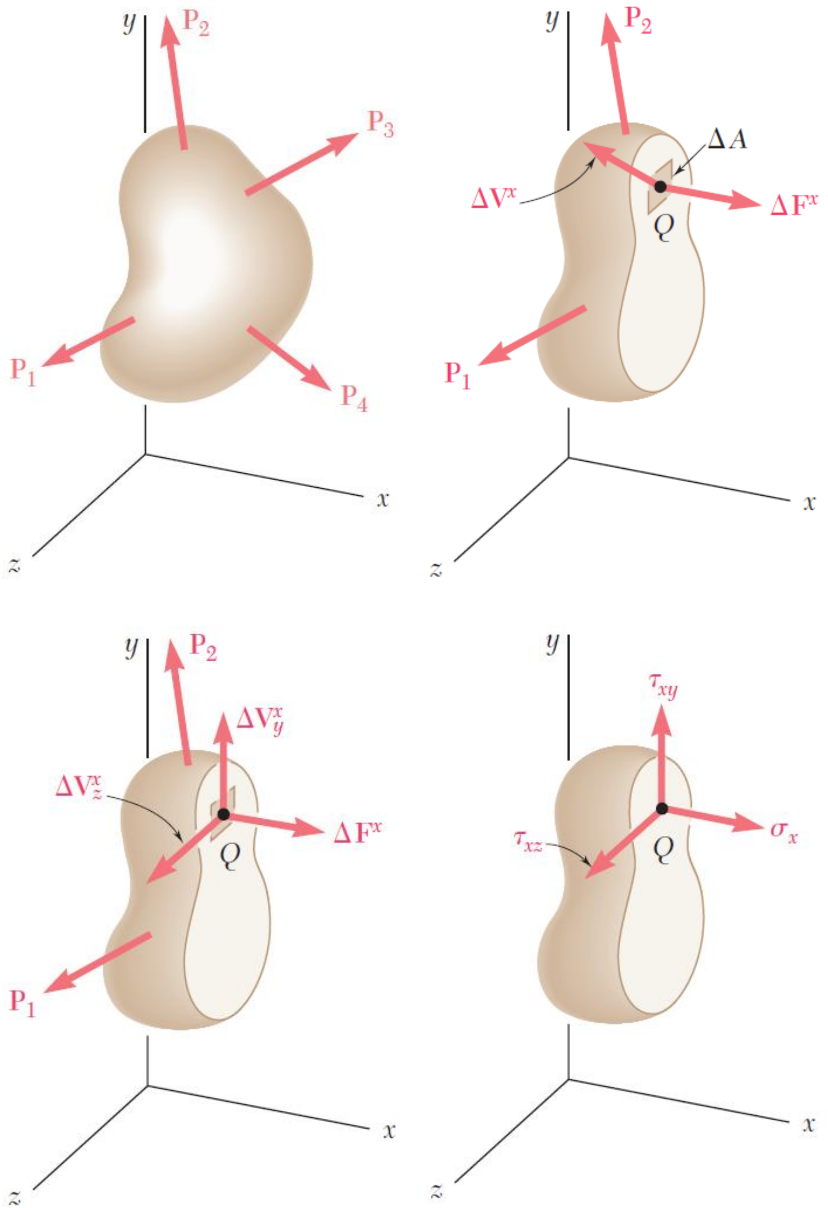
$$\sigma_{\alpha} = \sigma \cos^2 \alpha$$

$$\tau_{\alpha} = \frac{\sigma}{2} \sin 2\alpha$$

Examples of the Stress State of a Point



Stress Under General Loadings



- A member subjected to a general combination of loads is cut into two segments by a plane passing through a point of interest *Q*

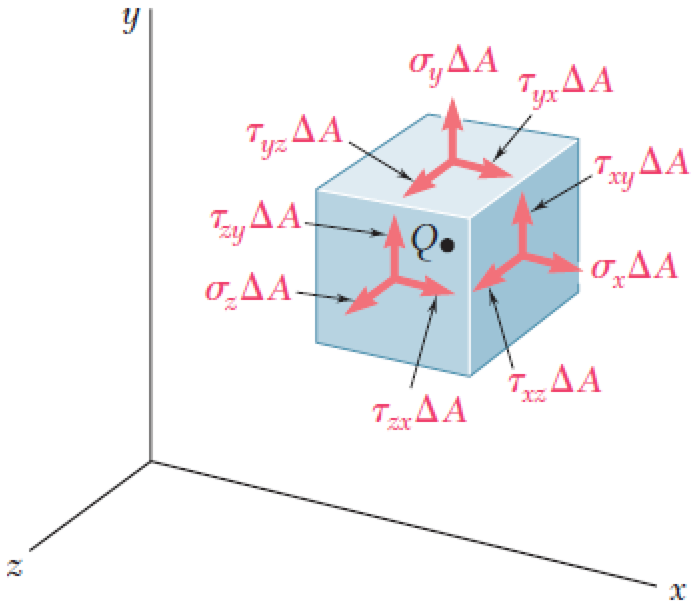
- The distribution of internal stress components may be defined as,

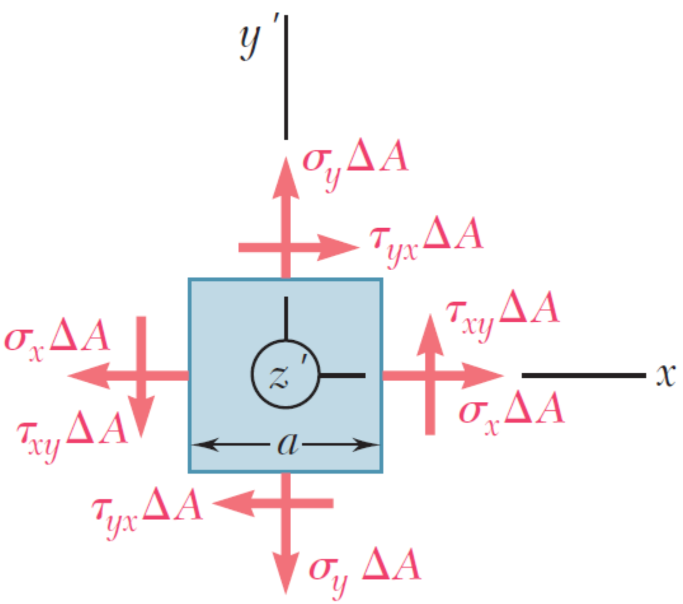
$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A}$$

- For equilibrium, an equal and opposite internal force and stress distribution must be exerted on the other segment of the member.

General Stress State of a Point





- Stress components are defined for the planes cut parallel to the *x*, *y* and *z* axes. For equilibrium, equal and opposite stresses are exerted on the hidden planes.
- The combination of forces generated by the stresses must satisfy the conditions for equilibrium:

$$\sum F_x = \sum F_y = \sum F_z = 0$$

$$\sum M_x = \sum M_y = \sum M_z = 0$$

- Consider the moments about the *z* axis:

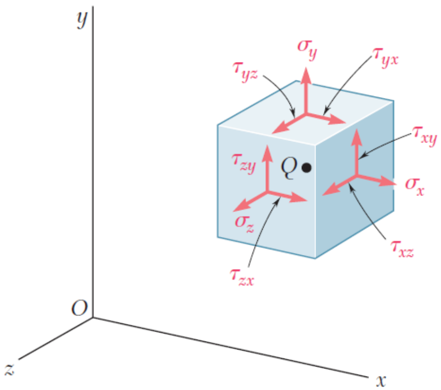
$$\sum M_z = 0 = (\tau_{xy} \Delta A)a - (\tau_{yx} \Delta A)a$$

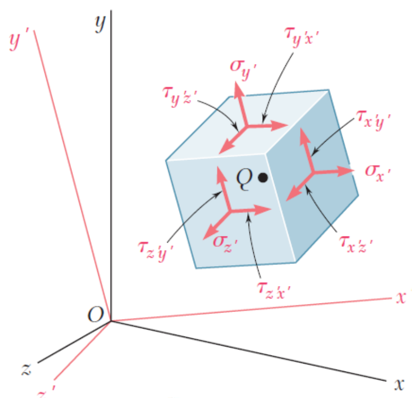
$$\tau_{xy} = \tau_{yx}$$

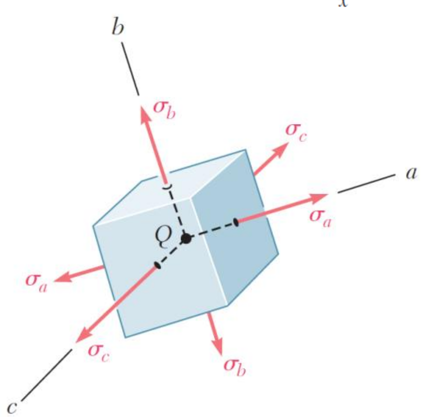
similarly, $\tau_{yz} = \tau_{zy}$ and $\tau_{yz} = \tau_{zy}$

- It follows that only 6 components of stress are required to define the complete state of stress

Principal Stress State of a Point







- The most general state of stress at a point may be represented by 6 components,

$\sigma_x, \sigma_y, \sigma_z$	normal stresses
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	shearing stresses
	(Note: $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)

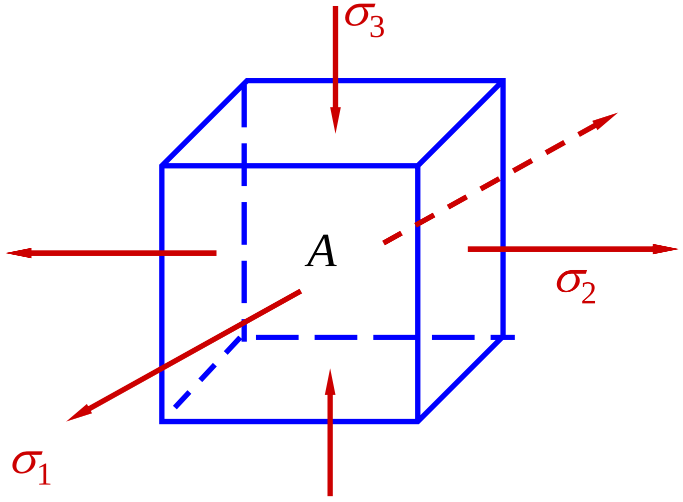
- Same state of stress is represented by a different set of components if axes are rotated.

- There must exist a unique orientation (Principal Directions) of the differential cube, having only normal stresses (Principal Stresses) acting on each of its six faces.

Ordering of Principal Stress State

$$\sigma_{\max} \geq \sigma_{\text{inter}} \geq \sigma_{\min}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$



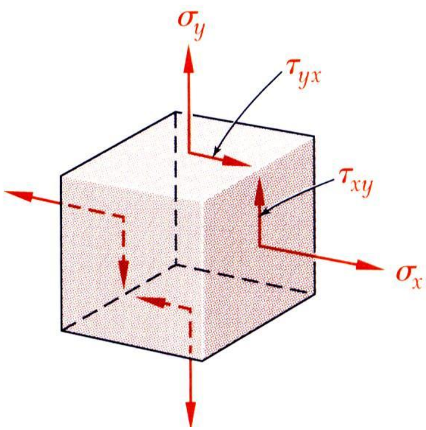
Principal axes & stresses

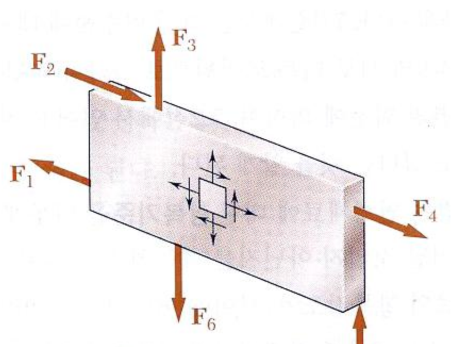
- Stress states classified based on the number of zero principal stresses:
  - Tri-axial stress state: none zero principal stresses;
  - Biaxial stress state: one zero principal stresses;
  - Uniaxial stress state: two zero principal stresses;

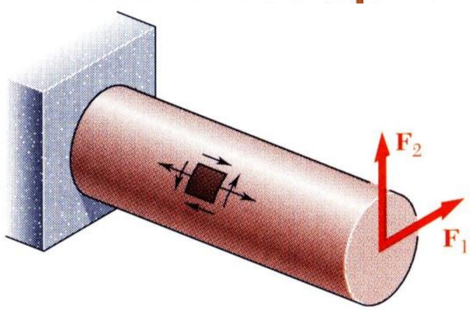
On the Damage Mechanism of Materials

- Damage (fracture) of brittle materials is governed by the maximum tensile stress
- Damage (yielding) of ductile materials is governed by the maximum shearing stress
- The goal of stress state analysis is to determine the surface/orientation on which normal/shearing stress achieves the maximum values.
- Strength analysis can be subsequently performed in terms of the maximum normal/shearing stresses.

Plane Stress States







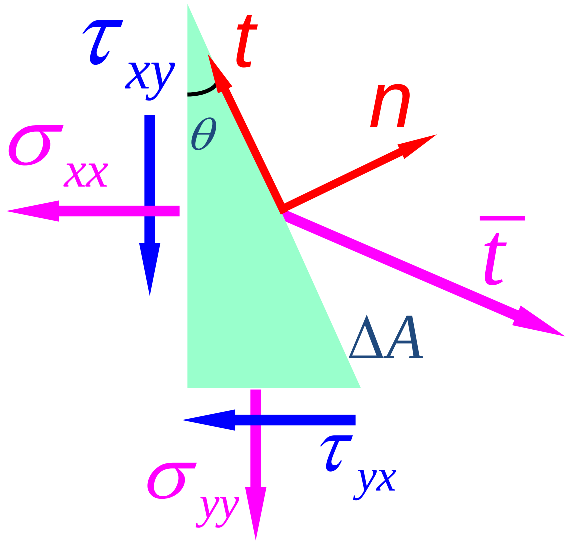
- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by

$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

- State of plane stress occurs in a thin plate subjected to forces acting in the mid-plane of the plate.

- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

2-D Cauchy's Relation



- Cauchy's relation:

$$\begin{cases} 0=\sum F_x \\ 0=\sum F_y \end{cases} \Rightarrow \begin{cases} t_x \Delta A = \sigma_{xx}\Delta A\cos\theta + \tau_{yx}\Delta A\sin\theta \\ t_y \Delta A = \tau_{xy}\Delta A\cos\theta + \sigma_{yy}\Delta A\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} t_x = \sigma_{xx}n_x + \tau_{yx}n_y \\ t_y = \tau_{xy}n_x + \sigma_{yy}n_y \end{cases}$$

$$\Rightarrow \begin{cases} t_\alpha = \sigma_{\beta\alpha}n_\beta = \sigma_{\alpha\beta}n_\beta \\ \mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\mathbf{n} \end{cases}$$

- Principal stresses & principal directions

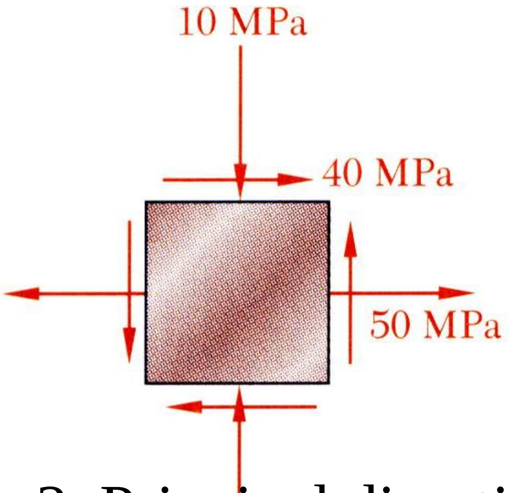
$$\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n} = \sigma\mathbf{n}$$

$$\Rightarrow \left(\boldsymbol{\sigma} - \sigma\,\mathrm{I}\right)\cdot\mathbf{n} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Sample Problem

- For the state of plane stress shown, determine the principal stresses and principal directions.



- Solution 1. Principal stresses

$$\begin{bmatrix} 50-\sigma & 40 \\ 40 & -10-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 50-\sigma & 40 \\ 40 & -10-\sigma \end{vmatrix} = 0 \Rightarrow (50-\sigma)(-10-\sigma) - 1600 = 0$$

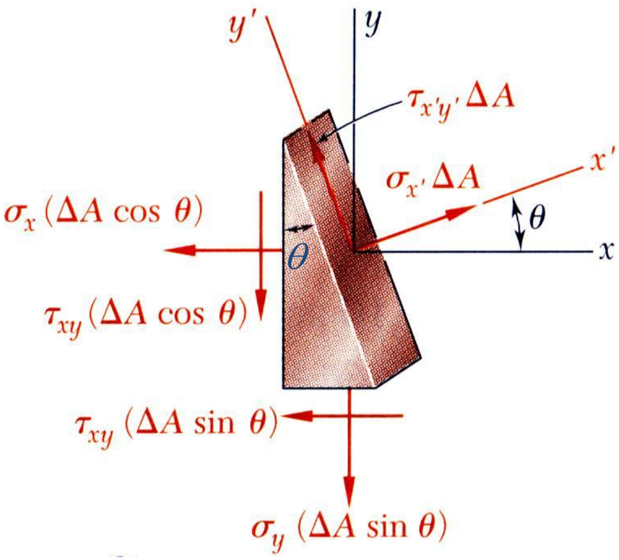
$$\Rightarrow \sigma^2 - 40\sigma - 2100 = 0 \Rightarrow \sigma_1 = 70, \sigma_2 = -30$$

2. Principal directions

$$\begin{bmatrix} 50-70 & 40 \\ 40 & -10-70 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} -20 & 40 \\ 40 & -80 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow n_1 = 2n_2 \Rightarrow \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{Bmatrix}$$

$$\begin{bmatrix} 50+30 & 40 \\ 40 & -10+30 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 80 & 40 \\ 40 & 20 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow n_2 = -2n_1 \Rightarrow \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{Bmatrix}$$

Plane Stress Transformation



- Consider the conditions for equilibrium of a prismatic element with axis perpendicular to the *x*, *y*, and *x'*.

$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta - \sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta - \sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta$$

- The equations may be rewritten to yield

$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\theta \to \theta + \pi/2\right)$$

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

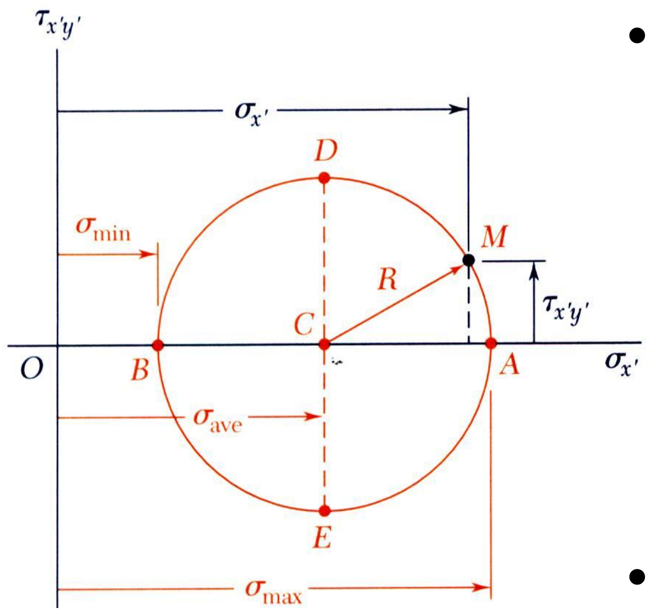
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

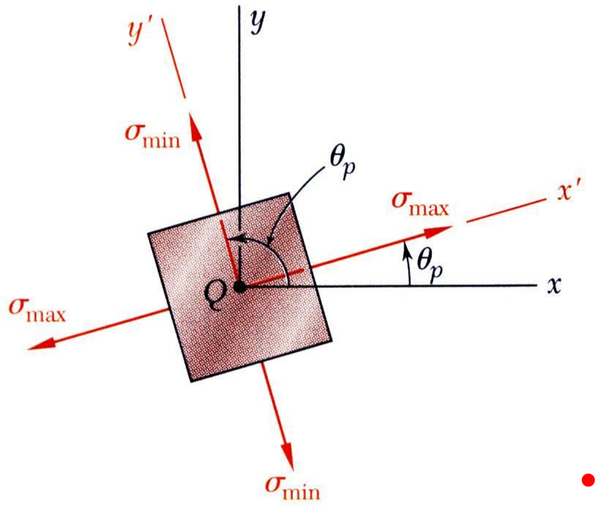
Plane Stress Transformation – Alternative Way

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{x'y'} & \sigma_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta & -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \\ -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta & \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{bmatrix}$$

Mohr's Circle





- The previous equations are combined to yield parametric equations for a circle,

$$\left.\begin{aligned}\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}&=\frac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta\\ \tau_{x'y'}&=-\frac{\sigma_x-\sigma_y}{2}\sin 2\theta+\tau_{xy}\cos 2\theta\end{aligned}\right\}\Rightarrow$$

$$\boxed{\left(\sigma_{x'}-\sigma_{ave}\right)^2+\tau_{x'y'}^2=R^2,\quad \sigma_{ave}=\frac{\sigma_x+\sigma_y}{2};\ R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

- Principal stresses

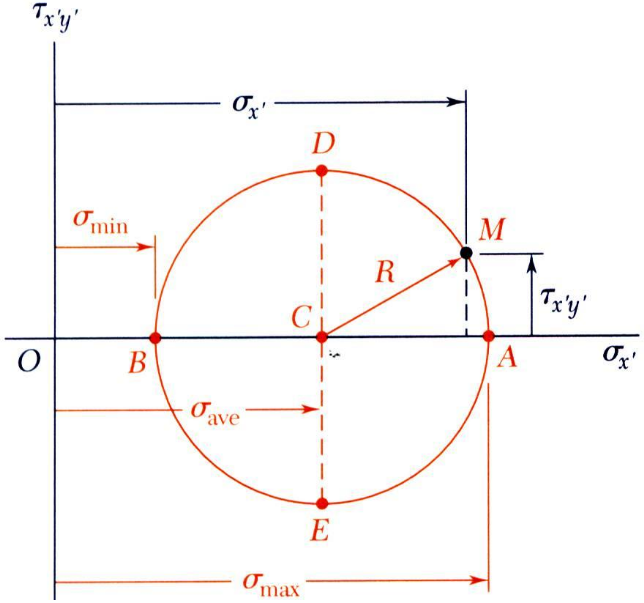
$$0=\tau_{x'y'}=-\frac{\sigma_x-\sigma_y}{2}\sin 2\theta+\tau_{xy}\cos 2\theta\Rightarrow$$

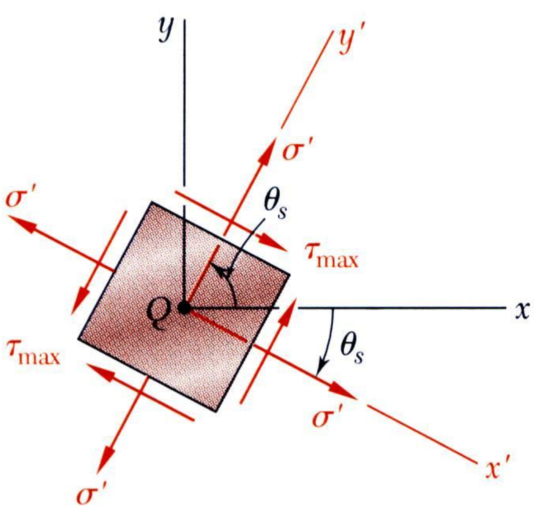
$$\begin{cases}\sigma_{x'}=\sigma_{ave}\pm R\quad\Rightarrow\sigma_{\max,\min}=\dfrac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\\ \tan 2\theta_p=\dfrac{\tau_{xy}}{\left(\sigma_x-\sigma_y\right)/2}\end{cases}$$

Note: defines two angles separated by 90°

- It is hard to predict which of the two angles results in the maximum/minimum principal stress.

Maximum Shearing Stress





Maximum shearing stress occurs for

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau_{x'y'}^2 = R^2$$

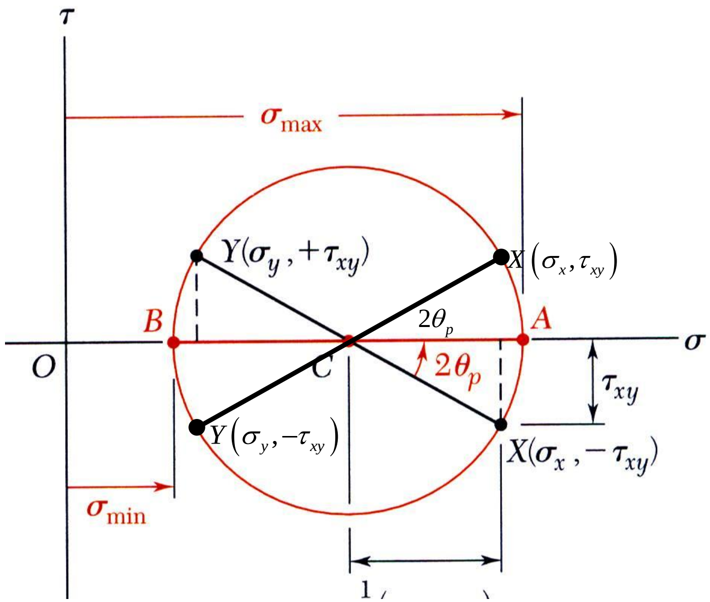
$$\sigma_{x'} = \sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$$

$$\Rightarrow \boxed{\begin{array}{l} \tau_{x'y'} = \tau_{\max} = R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \tan 2\theta_s = -\dfrac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \end{array}}$$

$$\tan 2\theta_s \tan 2\theta_p = -1$$

Note: defines two angles separated by 90° and offset from θ_p by 45°

Construction of Mohr's Circle



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau_{x'y'}^2 = R^2, \quad \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}; \ R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

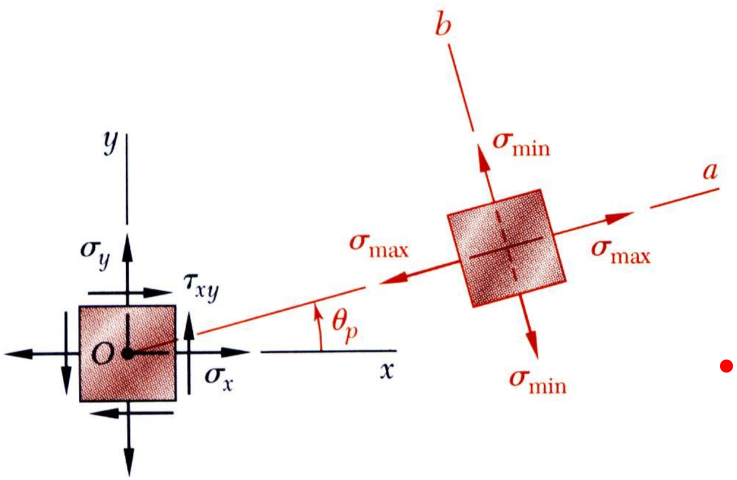
$$\sigma_{max,min} = \sigma_{ave} \pm R, \quad \tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

- Two options for plotting **the diameter *XY***

$$\begin{cases} \theta = 0 & \Rightarrow \sigma_{x'} = \sigma_x, \tau_{x'y'} = \tau_{xy} \\ \theta = \pi/2 & \Rightarrow \sigma_{x'} = \sigma_y, \tau_{x'y'} = -\tau_{xy} \end{cases}$$

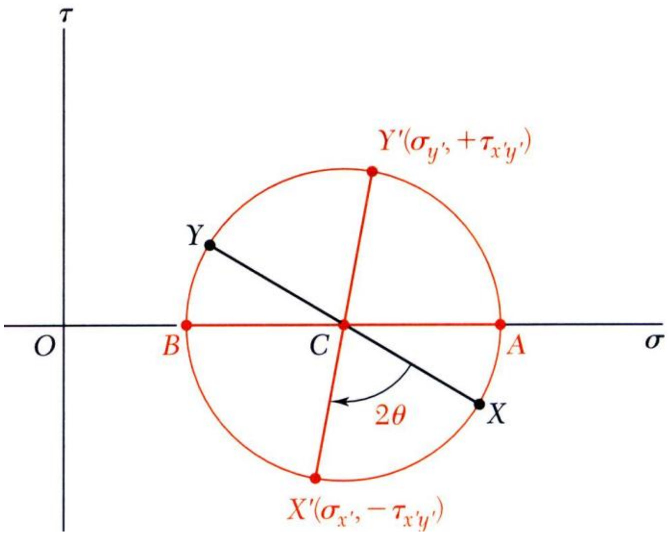
$$\left\{X\left(\sigma_x, \tau_{xy}\right), Y\left(\sigma_y, -\tau_{xy}\right)\right\}$$

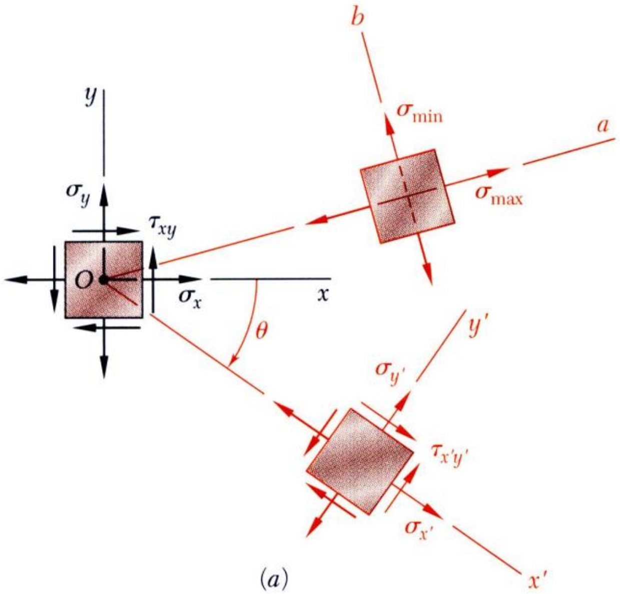
or $\left\{X\left(\sigma_x, -\tau_{xy}\right), Y\left(\sigma_y, \tau_{xy}\right)\right\}$



- **The second way is typically chosen, such that the direction of rotation of *Ox* to *Oa* is the same as *CX* to *CA*.**

Construction of Mohr's Circle



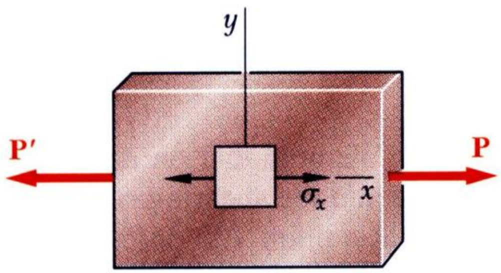


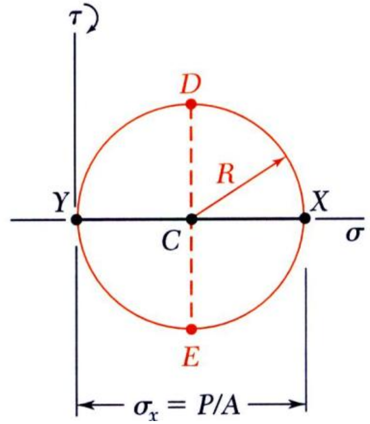
(a)

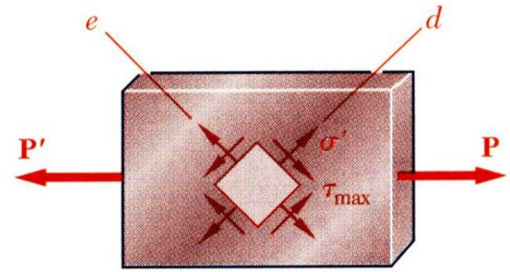
- With Mohr's circle uniquely defined, the state of stress at other axes orientations may be depicted.
- For the state of stress at an angle θ with respect to the *xy* axes, construct a new diameter *X'Y'* at an angle 2θ with respect to *XY*.
- Normal and shearing stresses are obtained from the coordinates *X'Y'*.

Construction of Mohr's Circle

- Mohr's circle for centric axial loading:



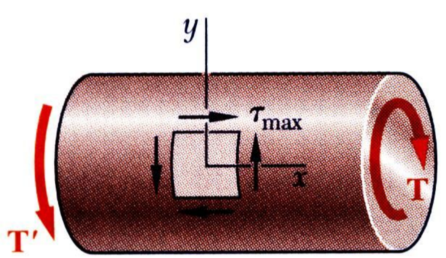


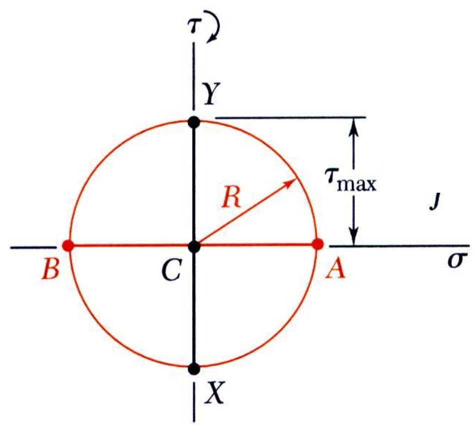


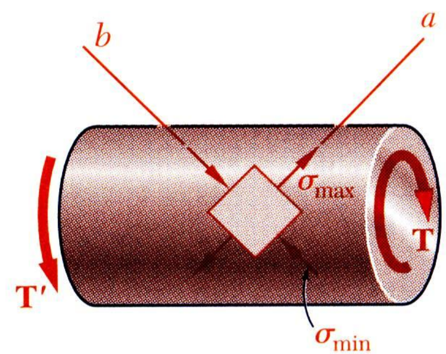
$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \tau_{xy} = 0$$

$$\sigma_x = \sigma_y = \tau_{xy} = \frac{P}{2A}$$

- Mohr's circle for torsional loading:



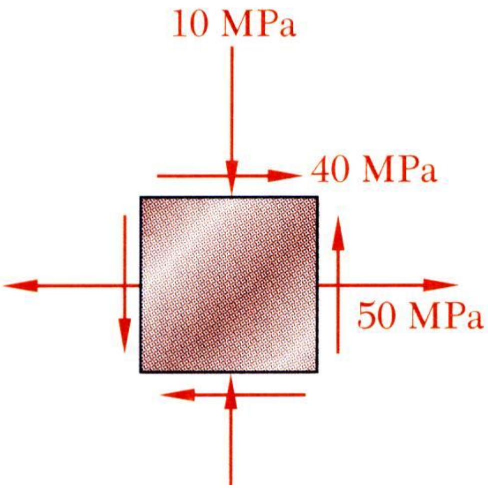




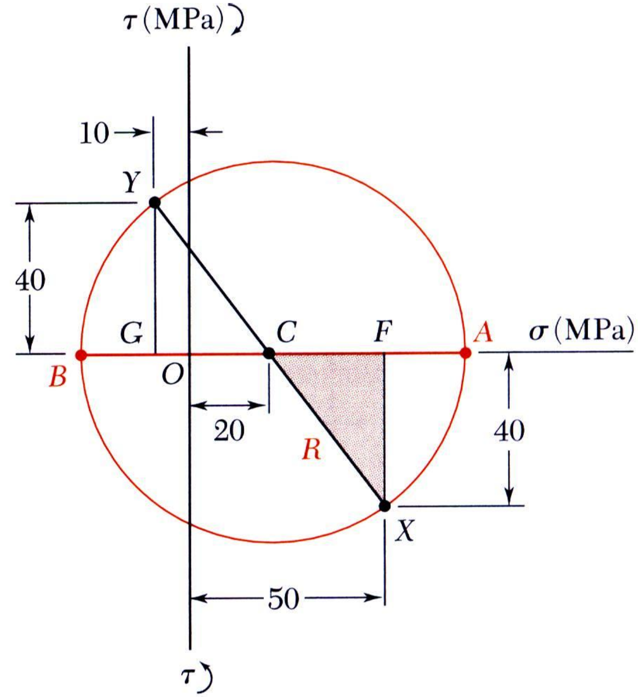
$$\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \frac{T}{W_P}$$

$$\sigma_x = \sigma_y = \frac{T}{W_P} \quad \tau_{xy} = 0$$

Sample Problem



For the state of plane stress shown, (a) construct Mohr's circle, determine (b) the principal planes, (c) the principal stresses, (d) the maximum shearing stress and the corresponding normal stress.



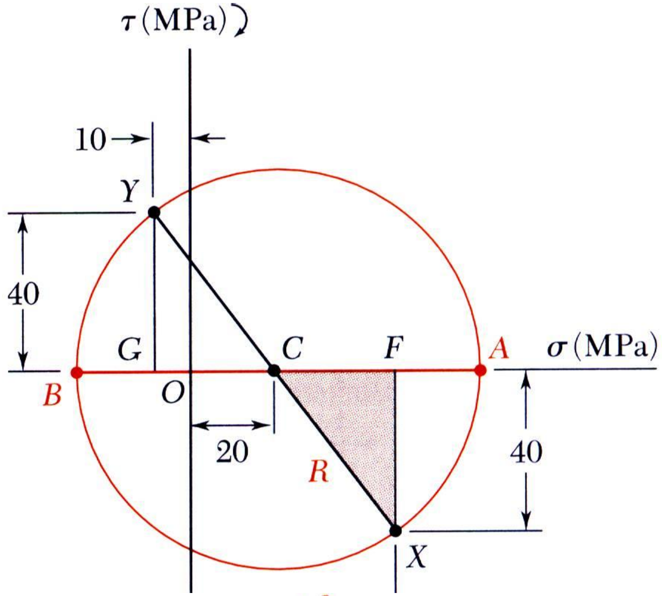
Solution:

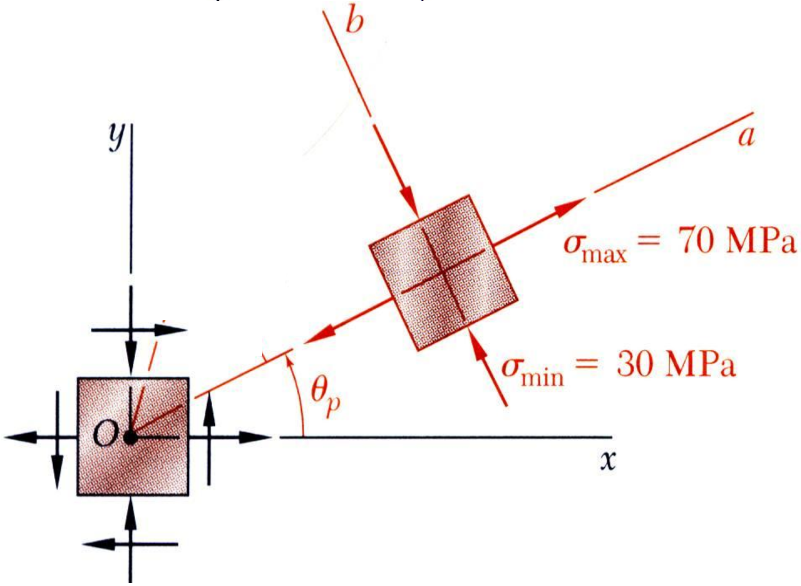
- Construction of Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50) + (-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$





- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\boxed{\sigma_{\max} = 70\,\text{MPa}}$$

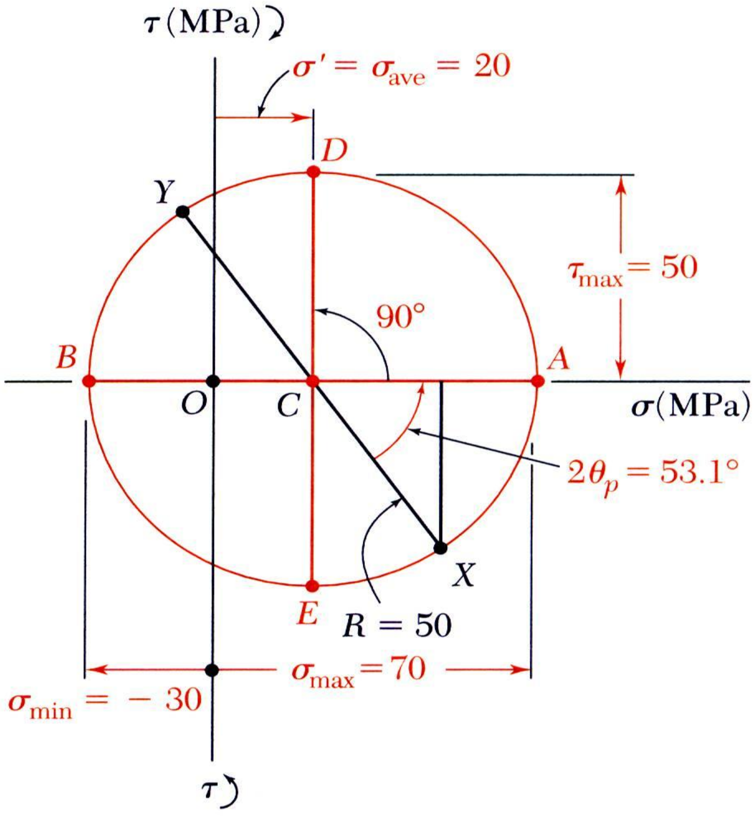
$$\sigma_{\max} = OB = OC - BC = 20 - 50$$

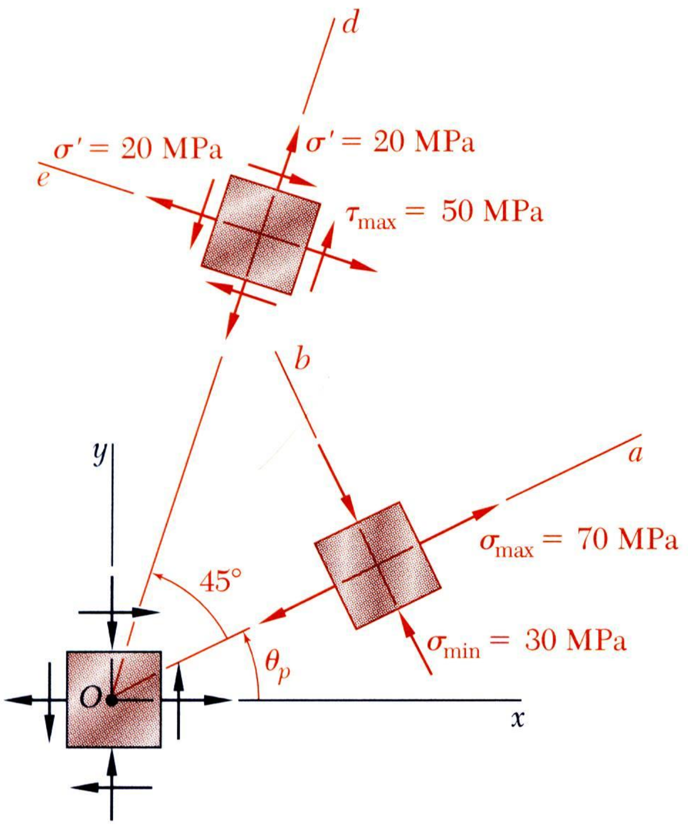
$$\boxed{\sigma_{\max} = -30\,\text{MPa}}$$

$$\tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30}$$

$$2\theta_p = 53.1°,\ 233.1°$$

$$\boxed{\theta_p = 26.6°,\ 116.6°}$$



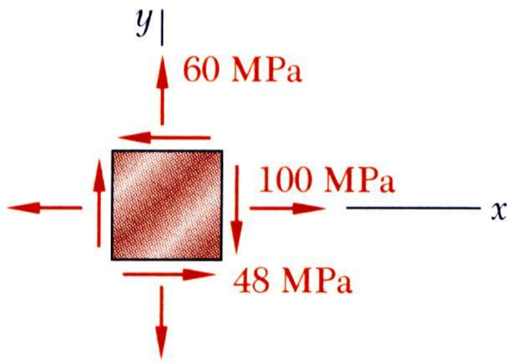


- Maximum shearing stress

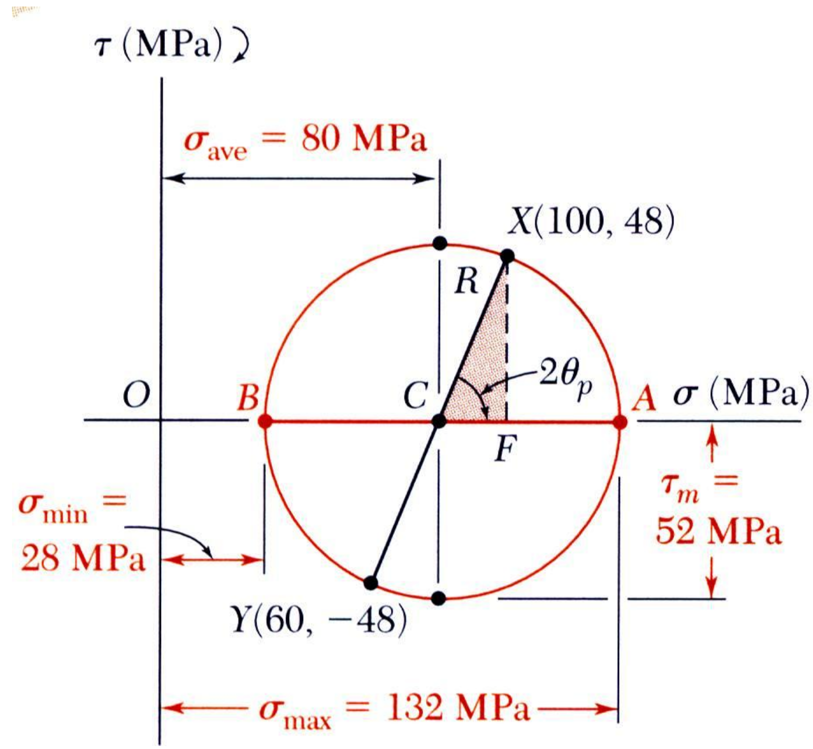
$$\theta_s = \theta_p + 45° \qquad \tau_{\max} = R \qquad \sigma' = \sigma_{ave}$$

$$\boxed{\theta_s = 71.6°,\ 161.6°} \qquad \boxed{\tau_{\max} = 50\ \text{MPa}} \qquad \boxed{\sigma' = 20\ \text{MPa}}$$

Sample Problem



For the state of stress shown, determine (a) the principal planes and the principal stresses, (b) the stress components exerted on the element obtained by rotating the given element counter-clockwise through 30 degrees.

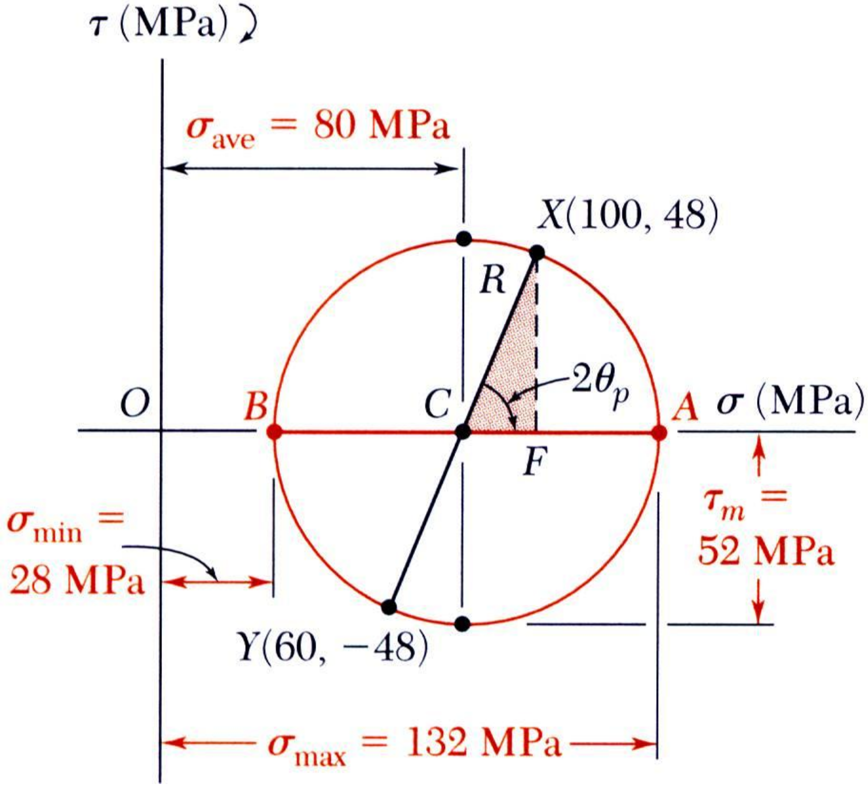


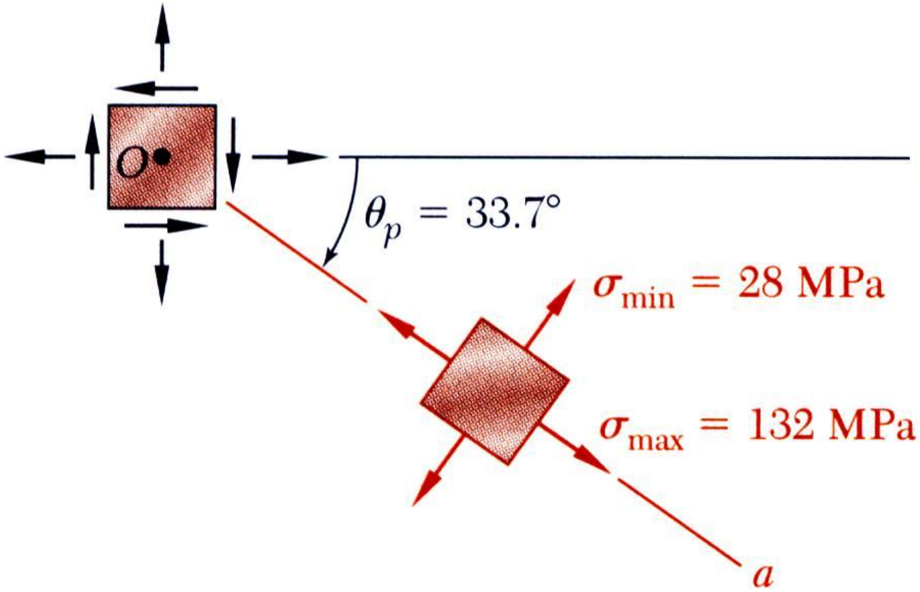
Solution:

- Construct Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 60}{2} = 80\,\text{MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52\,\text{MPa}$$





- Principal planes and stresses

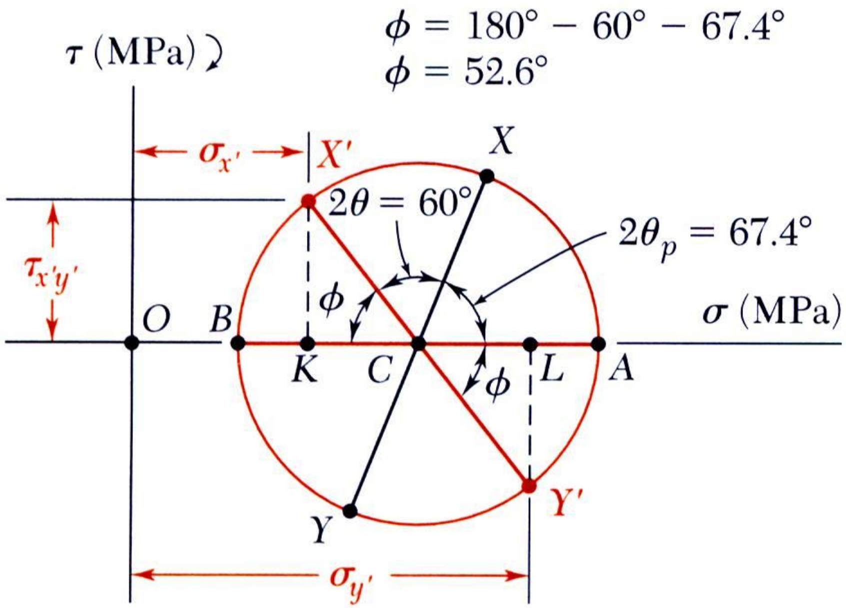
$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4$$

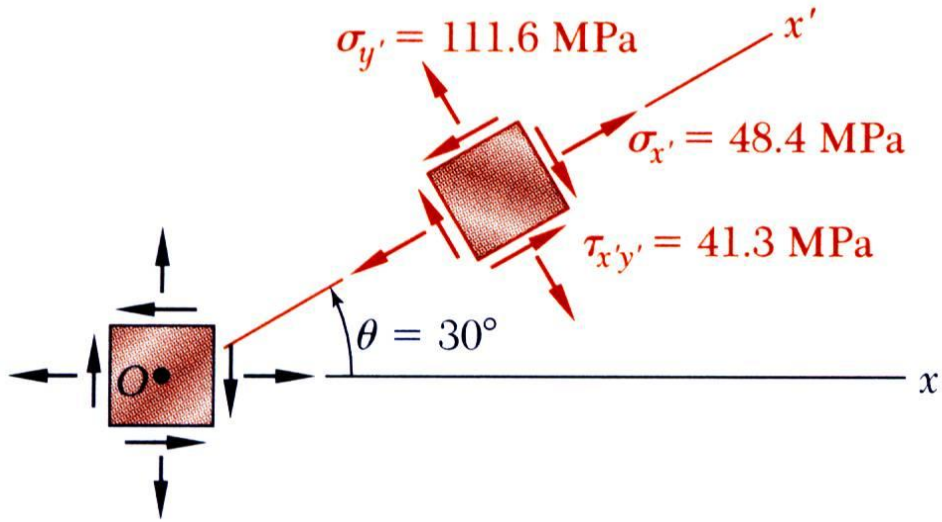
$$2\theta_p = 67.4°$$

$$\boxed{\theta_p = 33.7° \text{ clockwise}}$$

$$\sigma_{max} = OA = OC + CA$$
$$= 80 + 52$$
$$\boxed{\sigma_{max} = +132\,\text{MPa}}$$

$$\sigma_{max} = OA = OC - BC$$
$$= 80 - 52$$
$$\boxed{\sigma_{min} = +28\,\text{MPa}}$$





- Stress components after rotation by 30°
  Points *X'* and *Y'* on Mohr's circle that correspond to stress components on the rotated element are obtained by rotating *XY* counterclockwise through 60°

$$\phi = 180° - 60° - 67.4° = 52.6°$$

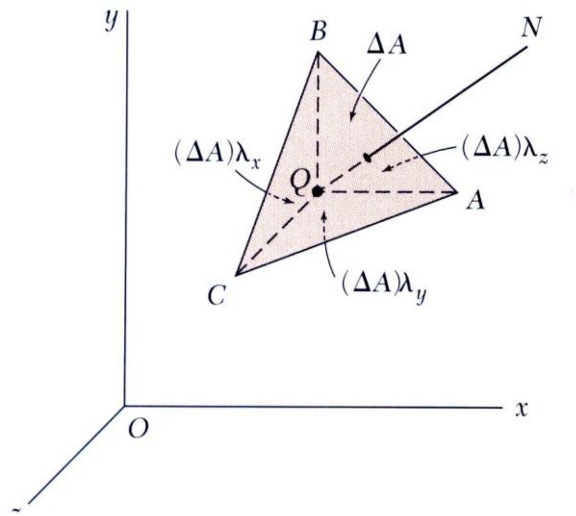
$$\sigma_{x'} = OK = OC - KC = 80 - 52\cos 52.6°$$

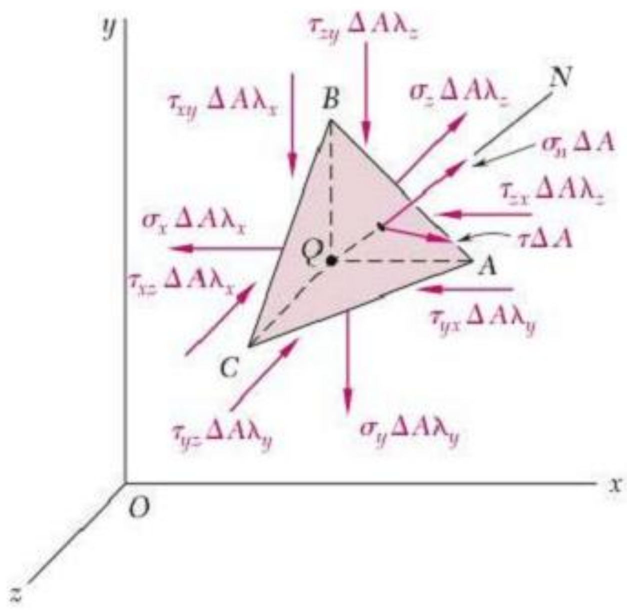
$$\sigma_{y'} = OL = OC + CL = 80 + 52\cos 52.6°$$

$$\tau_{x'y'} = KX' = 52\sin 52.6°$$

$$\sigma_{x'} = +48.4\,\text{MPa}$$
$$\sigma_{y'} = +111.6\,\text{MPa}$$
$$\tau_{x'y'} = 41.3\,\text{MPa}$$

3-D Cauchy's Relation





- Consider the general 3D state of stress at a point and the transformation of stress from element rotation
- State of stress at Q defined by:

  $$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$$

- Consider tetrahedron with face perpendicular to the line QN with direction cosines:

  $$\lambda_x = \frac{\bar{n} \cdot \bar{x}}{\|\bar{n}\| \, \|\bar{x}\|} = \cos(\bar{n}, \bar{x})$$

- Cauchy's relation:

$$\begin{cases} 0 = \sum F_x \\ 0 = \sum F_y \end{cases} \Rightarrow \begin{cases} t_x \Delta A = \sigma_{xx} \Delta A \lambda_x + \tau_{yx} \Delta A \lambda_y + \tau_{zx} \Delta A \lambda_z \\ t_y \Delta A = \tau_{xy} \Delta A \lambda_x + \sigma_{yy} \Delta A \lambda_y + \tau_{zy} \Delta A \lambda_z \\ t_z \Delta A = \tau_{xz} \Delta A \lambda_z + \tau_{yz} \Delta A \lambda_y + \sigma_{zz} \Delta A \lambda_z \end{cases}$$

$$\Rightarrow \begin{cases} t_i = \sigma_{ji} n_j = \sigma_{ij} n_j \\ \mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \mathbf{n} \end{cases}$$

3-D Cauchy's Relation

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n} = \sigma\mathbf{n} \Rightarrow$$

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \Rightarrow$$

$$\left\{\begin{aligned} &\sigma^3 - \sigma^2(\sigma_x + \sigma_y + \sigma_z) \\ &+\sigma(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2) \\ &-(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) \\ &= 0 \end{aligned}\right\}$$

$$\Rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

- Determine the normal and shearing stresses on the element in the *x*, *y*, and *z* direction of a given reference;
- Calculate the three stress invariants;
- Solve the cubic equation for its three roots (principal stresses);
- Substitute the roots back into original characteristic equation to solve for the principal directions.

I_1, I_2, and I_3 are known as **stress invariants** as they are independent of coordinate choices.

Sample Problem

- For the state of 3-D stress shown, determine the principal stresses and principal directions.

$$\tilde{\sigma} = \begin{bmatrix} 20 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

- Solution 1. Principal stresses

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 20 + 20 + 20 = 60$$

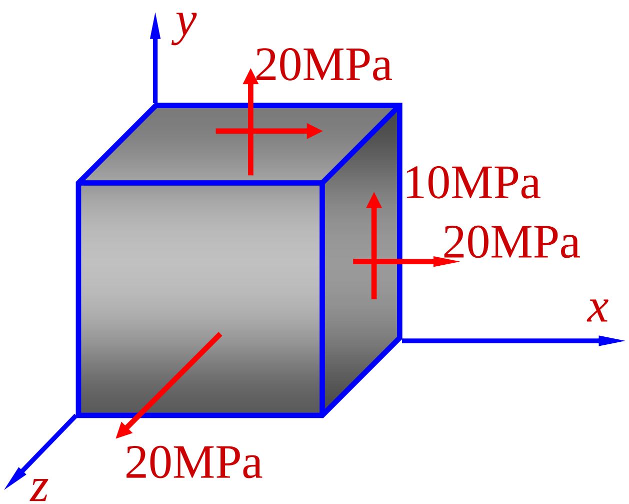
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = 400 + 400 + 400 - 100 = 1100$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = 8000 - 2000 = 6000$$

$$\Rightarrow \sigma^3 - 60\sigma^2 + 1100\sigma - 6000 = (\sigma - 20)(\sigma^2 - 40\sigma + 300) = 0$$

$$\Rightarrow (\sigma - 30)(\sigma - 20)(\sigma - 10) \approx 0$$

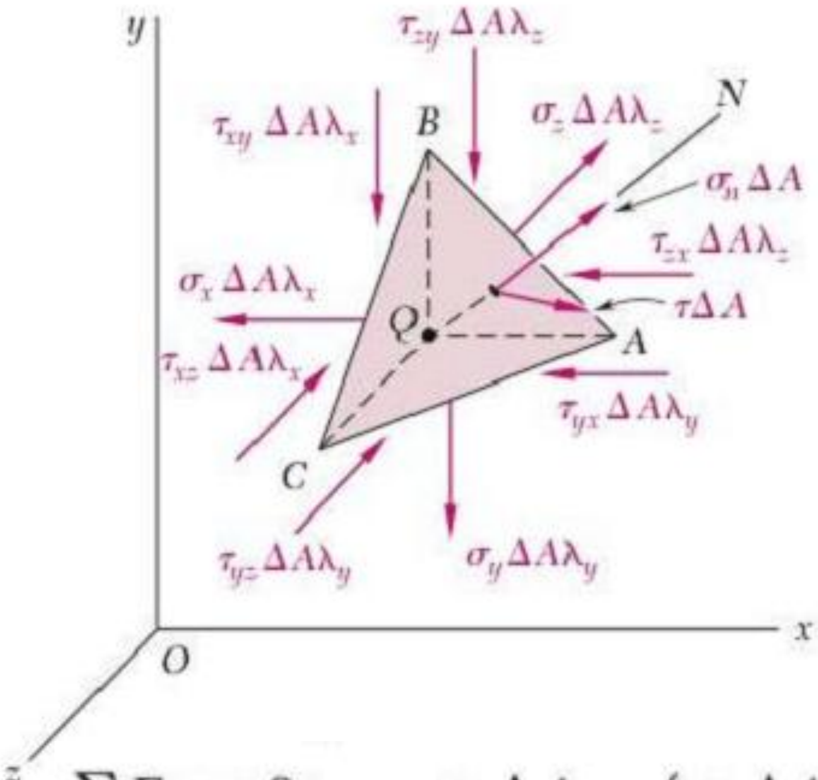
$$\Rightarrow \sigma_1 = 30; \sigma_1 = 20; \sigma_1 = 10$$



2. Principal directions

$$\begin{bmatrix} 20-\sigma & 10 & 0 \\ 10 & 20-\sigma & 0 \\ 0 & 0 & 20-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stress Transformation of General 3-D Stresses



- Consider the conditions for equilibrium of the tetrahedron generated from an oblique plane with the elemental volume.

- The requirement $\sum F_n = 0$ leads to

$$\sum F_n = 0: \quad \sigma_n\,\Delta A - (\sigma_x\,\Delta A\,\lambda_x)\lambda_x - (\tau_{xy}\,\Delta A\,\lambda_x)\lambda_y - (\tau_{xz}\,\Delta A\,\lambda_x)\lambda_z$$
$$-(\tau_{yx}\,\Delta A\,\lambda_y)\lambda_x - (\sigma_y\,\Delta A\,\lambda_y)\lambda_y - (\tau_{yz}\,\Delta A\,\lambda_y)\lambda_z$$
$$-(\tau_{zx}\,\Delta A\,\lambda_z)\lambda_x - (\tau_{zy}\,\Delta A\,\lambda_z)\lambda_y - (\sigma_z\,\Delta A\,\lambda_z)\lambda_z = 0$$

$$\Rightarrow \sigma_n = \sigma_x\lambda_x^2 + \sigma_y\lambda_y^2 + \sigma_z\lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x$$

Stress Transformation of General 3-D Stresses

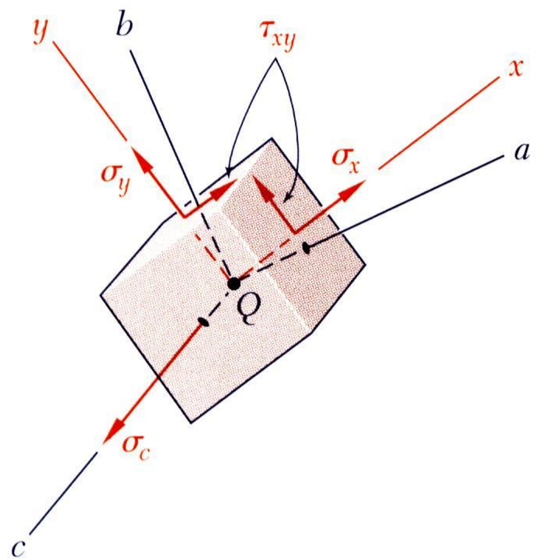
	x	y	z
n	λ_{nx}	λ_{ny}	λ_{nz}
o	λ_{ox}	λ_{oy}	λ_{oz}
p	λ_{px}	λ_{py}	λ_{pz}

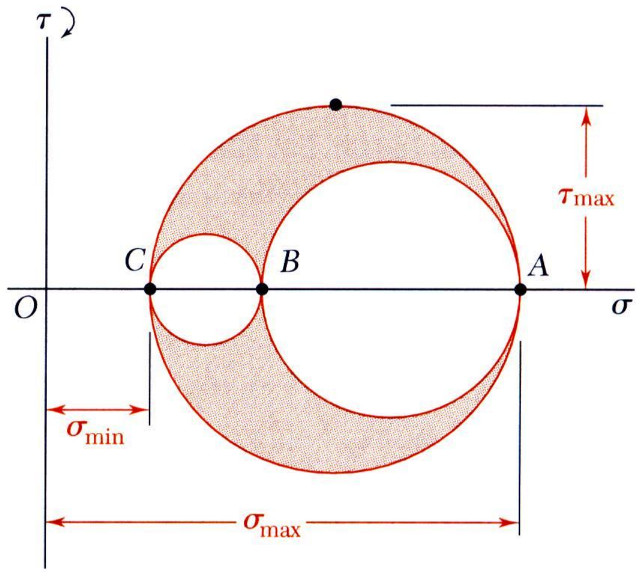
$$[a]=\begin{bmatrix} \lambda_{nx} & \lambda_{ny} & \lambda_{nz} \\ \lambda_{ox} & \lambda_{oy} & \lambda_{oz} \\ \lambda_{px} & \lambda_{py} & \lambda_{pz} \end{bmatrix}$$

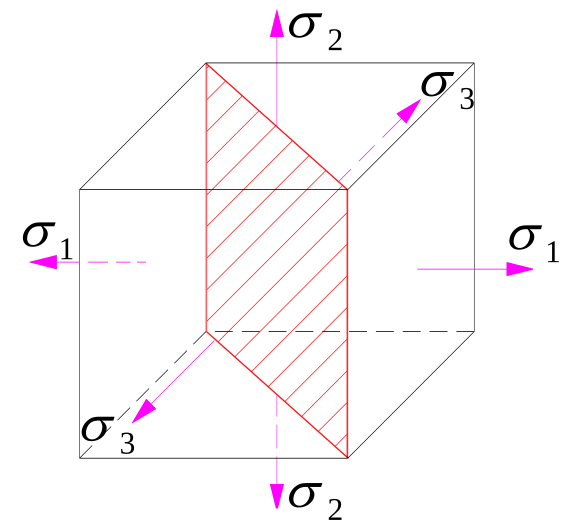
$$[\sigma']=[a][\sigma][a]^T=\begin{bmatrix} \lambda_{nx} & \lambda_{ny} & \lambda_{nz} \\ \lambda_{ox} & \lambda_{oy} & \lambda_{oz} \\ \lambda_{px} & \lambda_{py} & \lambda_{pz} \end{bmatrix}\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}\begin{bmatrix} \lambda_{nx} & \lambda_{ox} & \lambda_{px} \\ \lambda_{ny} & \lambda_{oy} & \lambda_{py} \\ \lambda_{nz} & \lambda_{oz} & \lambda_{pz} \end{bmatrix}$$

- Too complicated to find the principal stress states through stress transformation…

Application of Mohr's Circle in 3-D







- Points *A*, *B*, and *C* represent the principal stresses on the principal planes (shearing stress is zero)
- The three circles represent the normal and shearing stresses for rotation around each principal axis.
- Radius of the largest circle yields the maximum shearing stress.

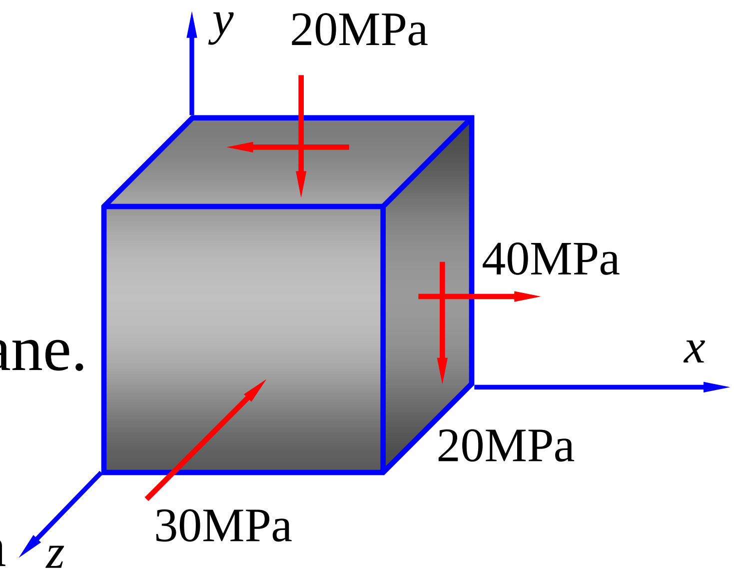
$$\tau_{max} = \frac{1}{2}\left|\sigma_{max} - \sigma_{min}\right|$$

Sample Problem

- For the state of 3-D stress shown below, determine the principal stresses and maximum shearing stress.
- Solution

1. Summarize the 3-D stress state.

$$[\sigma]=\begin{bmatrix} 40 & -20 & 0 \\ -20 & -20 & 0 \\ 0 & 0 & -30 \end{bmatrix}$$



2. Find the two principal stresses in *x*-*y* plane.

$$\left.\begin{matrix}\sigma_i \\ \sigma_j\end{matrix}\right\} = \frac{40-20}{2} \pm \sqrt{\left(\frac{40+20}{2}\right)^2 + 20^2} = \left\{\begin{matrix}46\text{ MPa} \\ -26\text{ MPa}\end{matrix}\right.$$

3. Rearrange the three principal stresses.

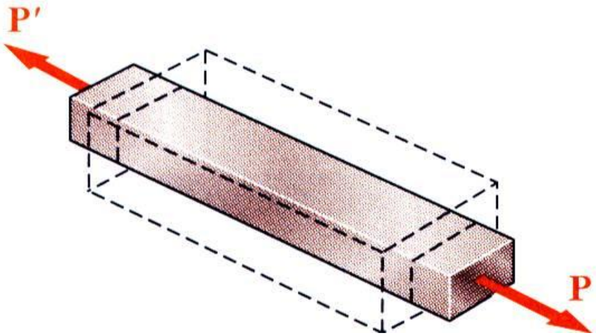
$$\sigma_1 = 46\text{ MPa},\ \sigma_2 = -26\text{ MPa},\ \sigma_3 = -30\text{ MPa}$$

4. Find the maximum shearing stress.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{46-(-30)}{2} = 38\text{ MPa}$$

Generalized Hooke's Law

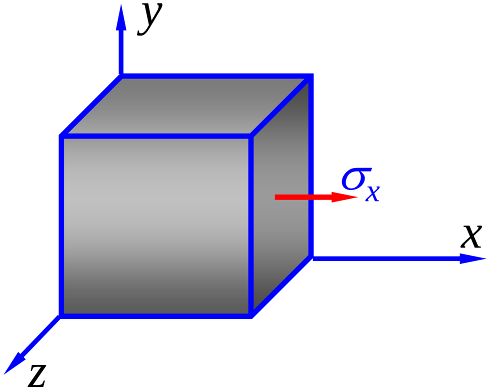
- For a bar under uniaxial tension:
- Axial strain: $\varepsilon = \sigma / E$
- Transverse strain: $\varepsilon' = -\nu\varepsilon = -\nu\sigma / E$



Uniaxial stress along x:

$$\varepsilon_x = \frac{\sigma_x}{E},$$

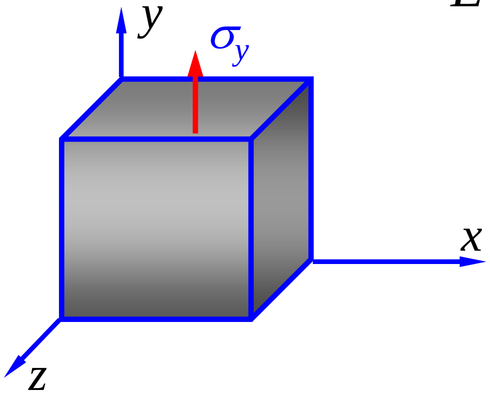
$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$



Along y:

$$\varepsilon_y = \frac{\sigma_y}{E},$$

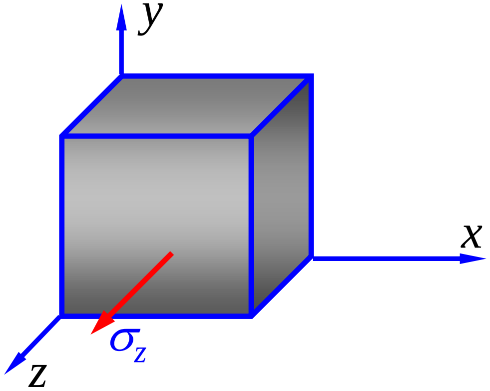
$$\varepsilon_z = \varepsilon_x = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$



Along z:

$$\varepsilon_z = \frac{\sigma_z}{E},$$

$$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E}$$

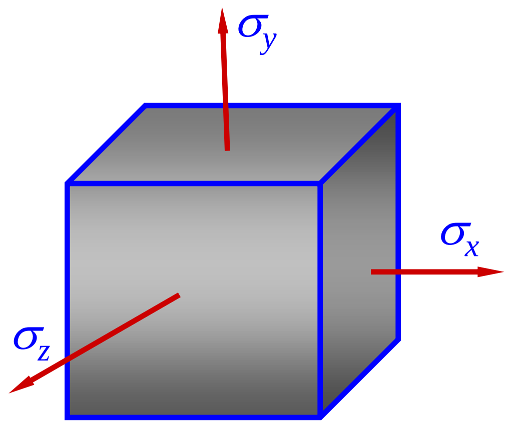


Generalized Hooke's Law

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{(1+\nu)}{E}\sigma_x - \frac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} = \frac{(1+\nu)}{E}\sigma_y - \frac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)$$

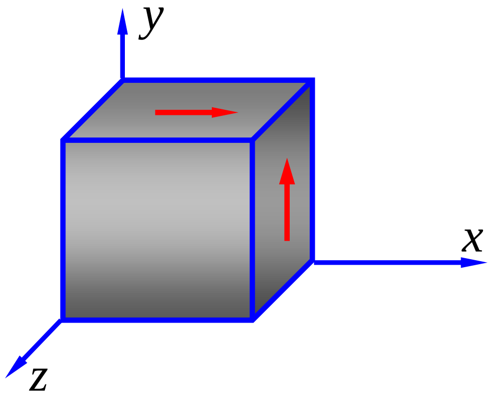
$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{(1+\nu)}{E}\sigma_z - \frac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)$$



Shearing Stresses

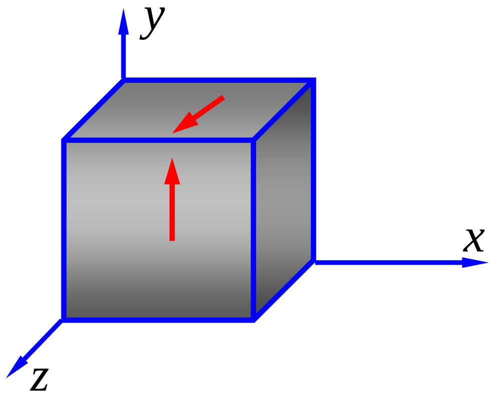
Shearing stress around z:

$$\tau_{xy} = G\gamma_{xy}$$



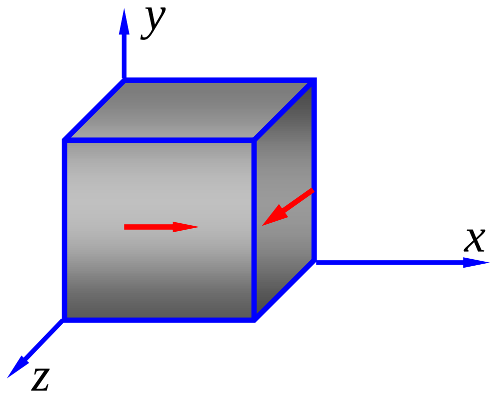
Around x:

$$\tau_{yz} = G\gamma_{yz}$$

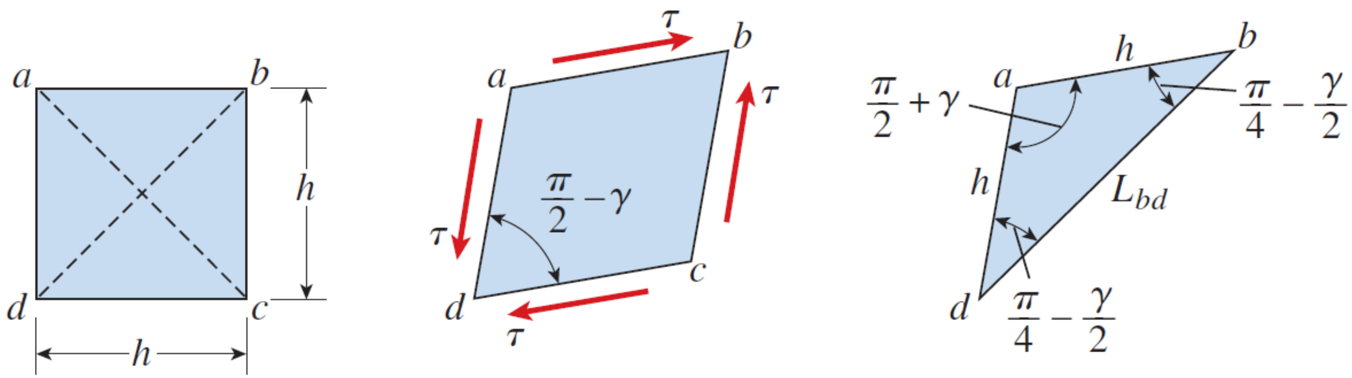


Around y:

$$\tau_{zx} = G\gamma_{zx}$$



The Relation among *E*, *ν* and *G*

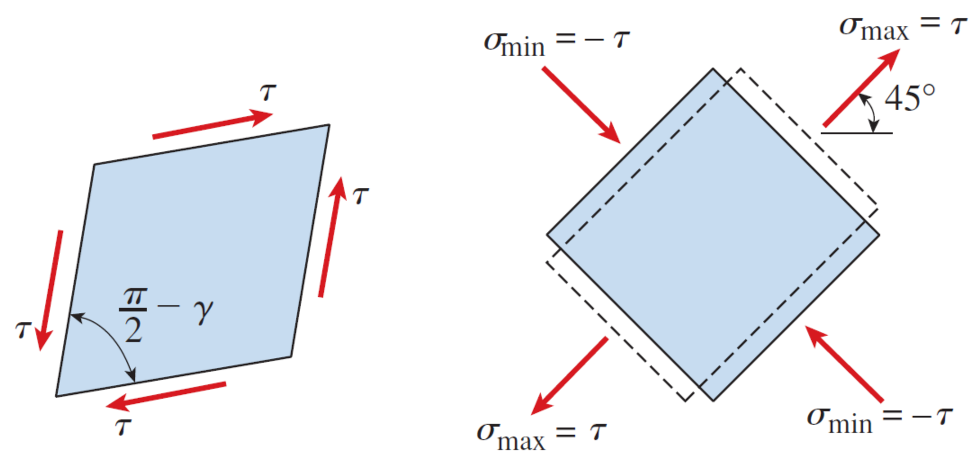


$$L_{bd} = h\sqrt{2}\left(1+\varepsilon\right)$$

$$L_{bd}^2 = h^2 + h^2 - 2h^2\cos\left(\pi/2+\gamma\right) = 2h^2\left(1+\sin\gamma\right)$$
$$\approx 2h^2\left(1+\gamma\right) = 2h^2\left(1+\tau/G\right)$$

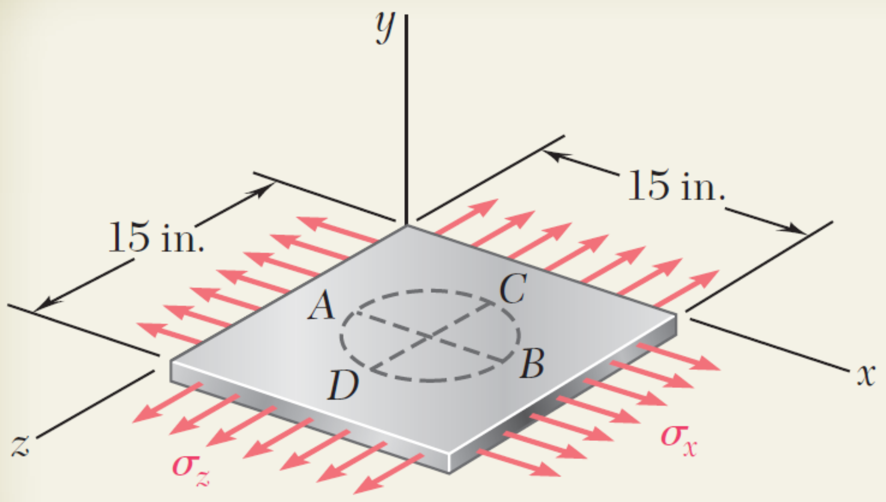
$$L_{bd}^2 = 2h^2\left(1+\varepsilon\right)^2 \approx 2h^2\left(1+2\varepsilon\right)$$
$$= 2h^2\left\{1+2\frac{\left(1+\nu\right)}{E}\sigma\right\}$$
$$= 2h^2\left\{1+2\frac{\left(1+\nu\right)}{E}\tau\right\}$$

$$\Rightarrow \boxed{E = 2G\left(1+\nu\right)}$$



Only two of the three elastic constants are independent.

Sample Problem



A circle of diameter d = 9 in. is scribed on an unstressed aluminum plate of thickness t = 3/4 in. Forces acting in the plane of the plate later cause normal stresses σ_x = 12 ksi and σ_z = 20 ksi.

For $E = 10\text{x}10^6$ psi and ν = 1/3, determine **the change** in:

a) the length of diameter *AB*,

b) the length of diameter *CD*,

c) the thickness of the plate, and

d) the volume of the plate.

SOLUTION:

- Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10\times10^6\,\text{psi}}\left[(12\,\text{ksi}) - 0 - \frac{1}{3}(20\,\text{ksi})\right]$$

$$= +0.533\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= -1.067\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= +1.600\times10^{-3}\,\text{in./in.}$$

- Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = (+0.533\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{B/A} = +4.8\times10^{-3}\,\text{in.}}$$

$$\delta_{C/D} = \varepsilon_z d = (+1.600\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{C/D} = +14.4\times10^{-3}\,\text{in.}}$$

$$\delta_t = \varepsilon_y t = (-1.067\times10^{-3}\,\text{in./in.})(0.75\,\text{in.})$$

$$\boxed{\delta_t = -0.800\times10^{-3}\,\text{in.}}$$

- Find the change in volume

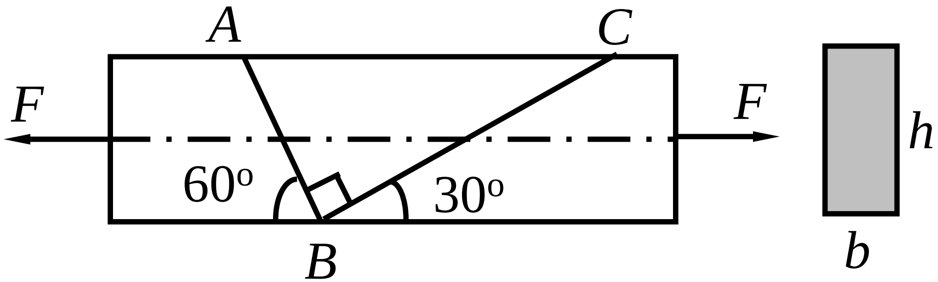
$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times10^{-3}\,\text{in}^3/\text{in}^3$$

$$\Delta V = eV = 1.067\times10^{-3}(15\times15\times0.75)\,\text{in}^3$$

$$\boxed{\Delta V = +0.187\,\text{in}^3}$$

Sample Problem

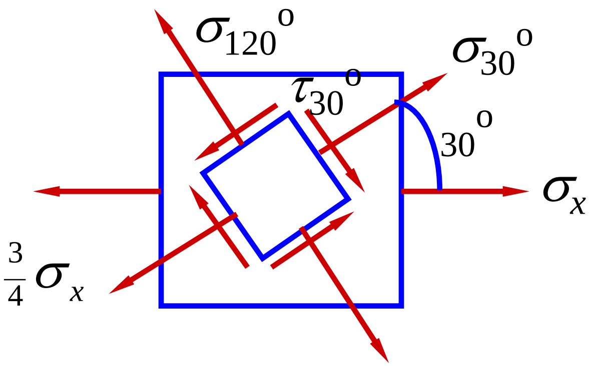
- For the bar shown, find the change in length of line segment *BC* and in right angle *ABC*. Assume the Young's modulus and Poisson's ratio as *E* and *v* respectively.
- Solution



1. Stress state in reference coordinate system

$$\sigma_x = F/bh$$

All other stress components vanish.



2. Stress state in transformed coordinates:

$$\sigma_{30^0} = \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \tfrac{3}{4}\sigma_x$$

$$\tau_{30^0} = \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\tfrac{\sqrt{3}}{4}\sigma_x$$

$$\sigma_{120^0} = \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta = \tfrac{1}{4}\sigma_x$$

3. Longitudinal strain along *BC*

$$\varepsilon_{30^0} = \frac{1}{E}\left(\sigma_{30^0} - \nu\sigma_{120^0}\right) = \frac{3-\nu}{4E}\sigma_x = \frac{(3-\nu)F}{4Ebh}$$

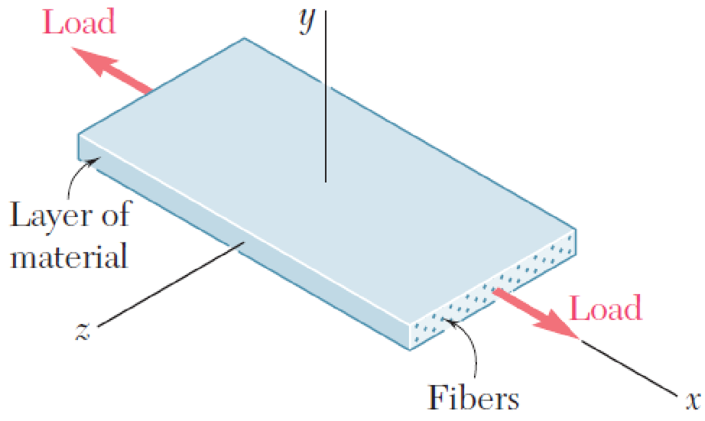
4. The change in length of *BC*

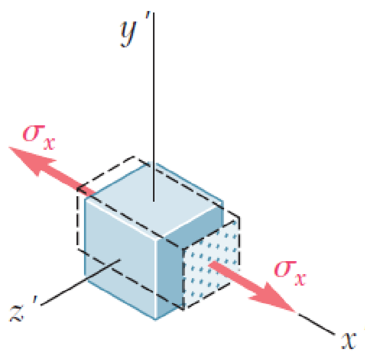
$$\Delta l_{BC} = \varepsilon_{30^0} \times l_{BC} = 2h\varepsilon_{30^0} = \frac{(3-\nu)F}{2Eb}$$

5. The change in right angle *ABC*

$$\gamma = \frac{\tau_{30^0}}{G} = \frac{\frac{\sqrt{3}}{4}\sigma_x}{\frac{E}{2(1+\nu)}} = \frac{\sqrt{3}(1+\nu)}{2E}\sigma_x = \frac{\sqrt{3}}{2}\frac{(1+\nu)F}{bhE}$$

Fiber-reinforced Composites (Anisotropy)





- *Fiber-reinforced composite materials* are formed from *lamina* of fibers of graphite, glass, or polymers embedded in a resin matrix.

- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

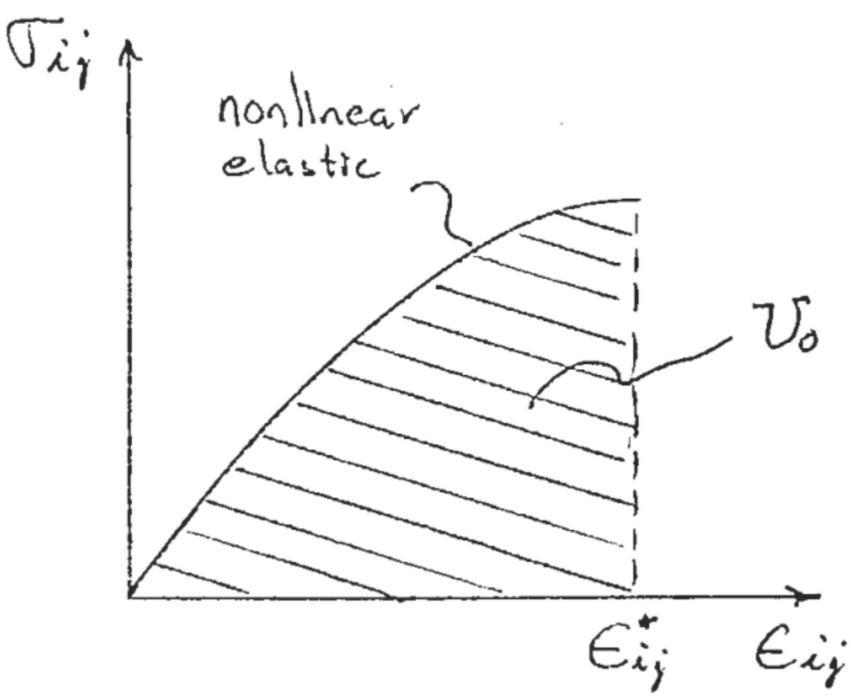
$$E_x = \frac{\sigma_x}{\varepsilon_x} \quad E_y = \frac{\sigma_y}{\varepsilon_y} \quad E_z = \frac{\sigma_z}{\varepsilon_z}$$

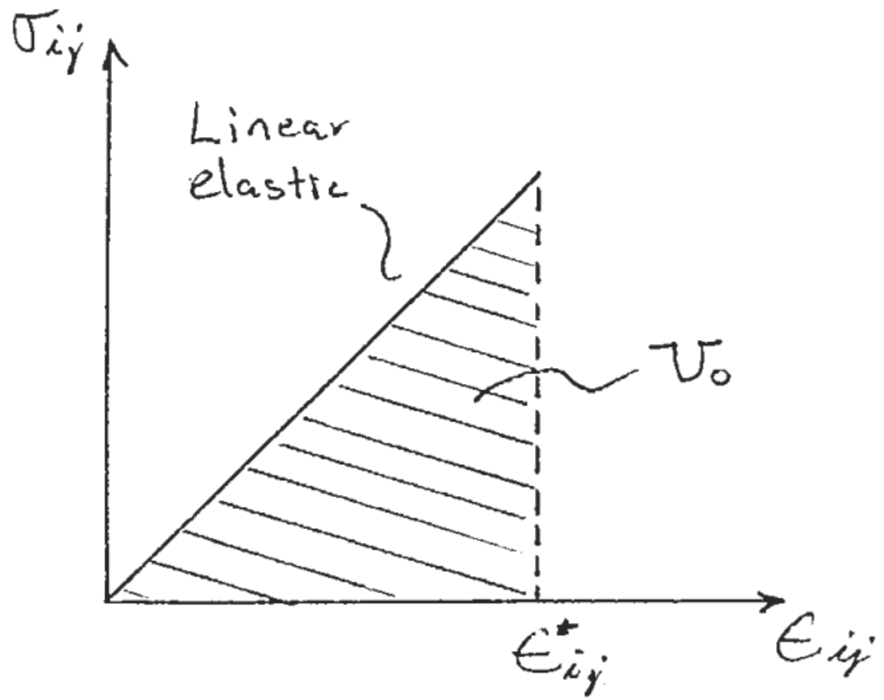
- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz} = -\frac{\varepsilon_z}{\varepsilon_x}$$

- Materials with directionally dependent mechanical properties are *anisotropic*.

Strain Energy under Generalized Stress States





- Strain energy density of non-linearly elastic material under uniaxial stress state

$$du = \sigma d\varepsilon$$

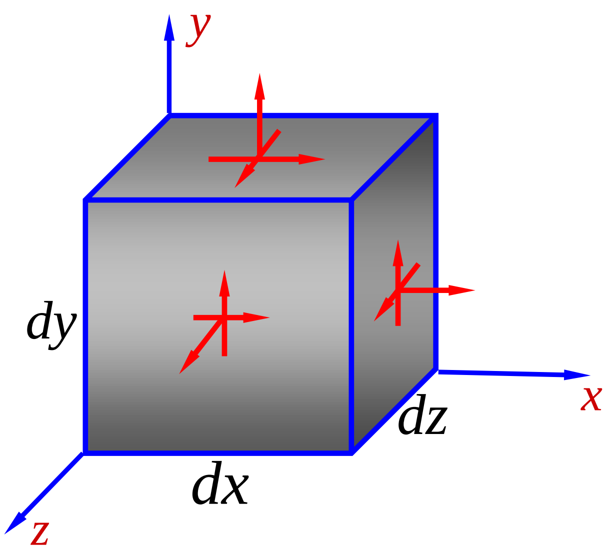
- Strain energy density of linearly elastic material under uniaxial stress state

$$u = \frac{1}{2}\sigma\varepsilon$$

Strain Energy under Generalized Stress States

- Strain energy density of non-linearly elastic material under generalized 3-D stress states

$$du = \sigma_{ij} d\varepsilon_{ij}$$
$$= \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z$$
$$+ \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx}$$



- Strain energy density of linearly elastic material under generalized 3-D stress states

$$U = \frac{1}{2}\begin{bmatrix} \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z \\ + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} \end{bmatrix}$$
$$= \frac{1}{2E}\begin{pmatrix} \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \\ 2\nu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) \end{pmatrix}$$
$$+ \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$$

Volume Strain, Mean Stress and Bulk Modulus

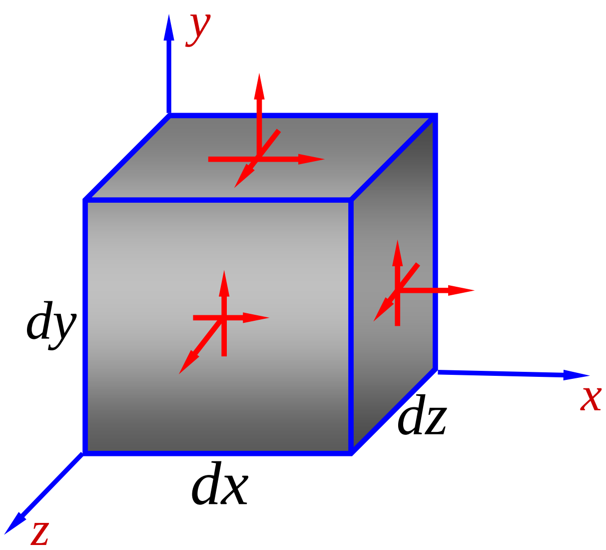
- Volume strain

$V_0 = dxdydz$

$V_1 = dx(1+\varepsilon_x)dy(1+\varepsilon_y)dz(1+\varepsilon_z)$

$= dxdydz\left[1+\varepsilon_x+\varepsilon_y+\varepsilon_z+O\left(\varepsilon^2\right)\right]$

$$\Rightarrow \varepsilon_V = \frac{V_1 - V_0}{V_0} \approx \varepsilon_x+\varepsilon_y+\varepsilon_z = \frac{(1\text{-}2\nu)}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)$$



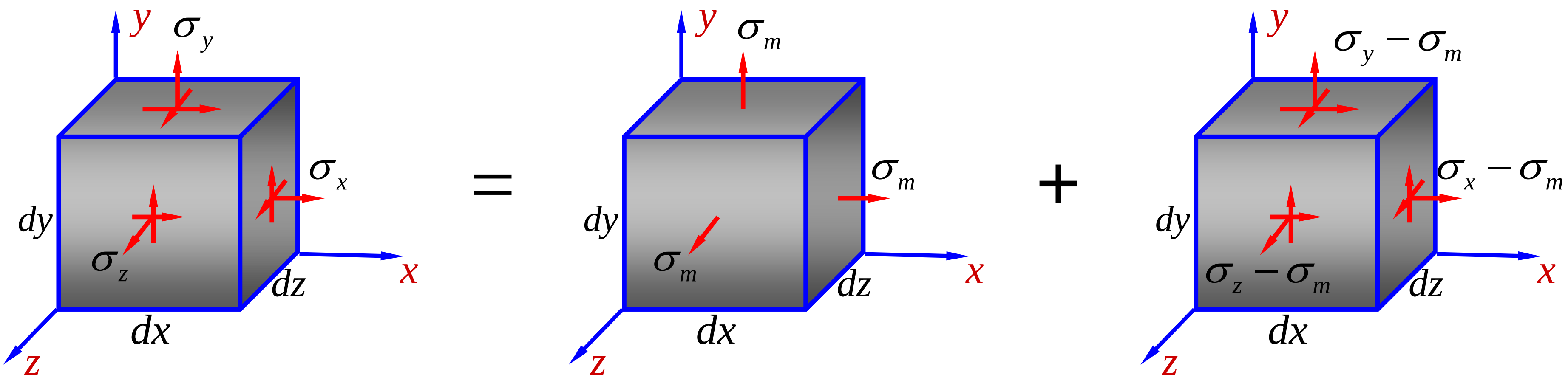
- For a given stress state, if no volume change happen, the first invariant of the stress tensor must be zero.

$$\sigma_x+\sigma_y+\sigma_z=0$$

- Bulk modulus can be defined as the ratio between mean stress and volume strain

$$K=\sigma_m/\varepsilon_V=\left(\sigma_x+\sigma_y+\sigma_z\right)/3\varepsilon_V=\frac{E}{3(1-2\nu)} \quad \left(0<\nu<1/2\right)$$

Volumetric & Distortion Energy Density



$$u = u_V + u_d$$

(a) Mean stress tensor

(b) Deviatoric stress tensor

- Volumetric energy density:

$$u_V = \frac{1}{2}\sigma_m\varepsilon_V = \frac{3(1-2\nu)}{2E}\sigma_m{}^2 = \frac{(1-2\nu)}{6E}\left(\sigma_x+\sigma_y+\sigma_z\right)^2$$

- Distortion energy density:

$$u_d = \frac{(1+\nu)}{6E}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+\left(\sigma_z-\sigma_x\right)^2\right]+\frac{1}{2\mathrm{G}}\left(\tau_{\mathrm{xy}}{}^2+\tau_{\mathrm{yz}}{}^2+\tau_{\mathrm{zx}}{}^2\right)$$

Contents

- The Stress State of a Point（点的应力状态）
- Simple, General & Principal Stress State（简单、一般和主应力状态）
- Ordering of Principal Stresses（主应力排序）
- On the Damage Mechanisms of Materials（材料失效机制）
- Plane Stress States（平面应力）
- 2-D Cauchy’s Relation（平面柯西关系）
- Plane Stress Transformation（平面应力转换）
- Mohr’s Circle（莫尔圆）
- Maximum Shearing Stress（最大切应力）
- Construction of Mohr’s Circle （莫尔圆的构建）

Contents

- 3-D Cauchy's Relation（三维柯西关系）
- Stress Transformation of General 3-D Stresses（空间应力转换）
- Application of Mohr's Circle in 3-D（莫尔圆在三维应力状态中的应用）
- Generalized Hooke's Law（广义胡克定律）
- The Relation among *E*, *v* and *G*（杨氏模量、泊松系数及剪切模量间的关系）
- Fiber-reinforced Composites（纤维增强复合材料）
- Strain Energy Density under Generalized Stress States（一般应力状态下的应变能）
- Volume Strain, Mean Stress and Bulk Modulus（体应变、平均应力和体积模量）
- Volumetric & Distortion Energy Density（体积应变能密度和畸变能密度）