Stress Measures

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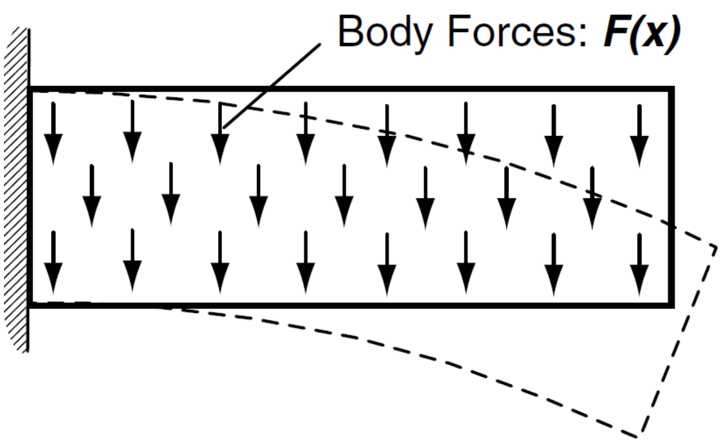
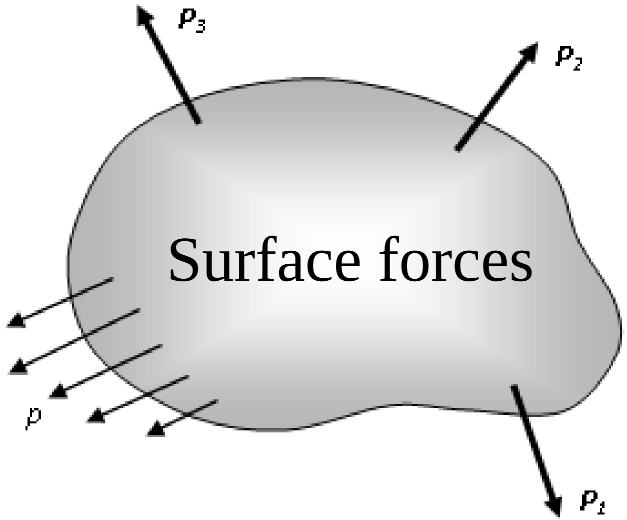
Outline

- Body and surface forces（体力与面力）
- Traction/stress vector（应力矢量）
- Cauchy stress tensor （柯西应力张量）
- Traction on oblique planes（斜面上的应力）
- Different stress measures（应力度量）
- Stress measures for infinitesimal deformation（小变形应力度量）
- Principal stresses and directions（主应力与主方向）
- Octahedral stresses（八面体应力）
- Hydrostatic, deviatoric and von Mises effective stresses（平均应力、偏应力、米泽斯等效应力）
- Conservation of linear momentum（线动量守恒）
- Conservation of angular momentum（角动量守恒）
- Rate of work done by stresses（应力做功的速率）

Body and Surface Forces

- External loads include body and surface forces.





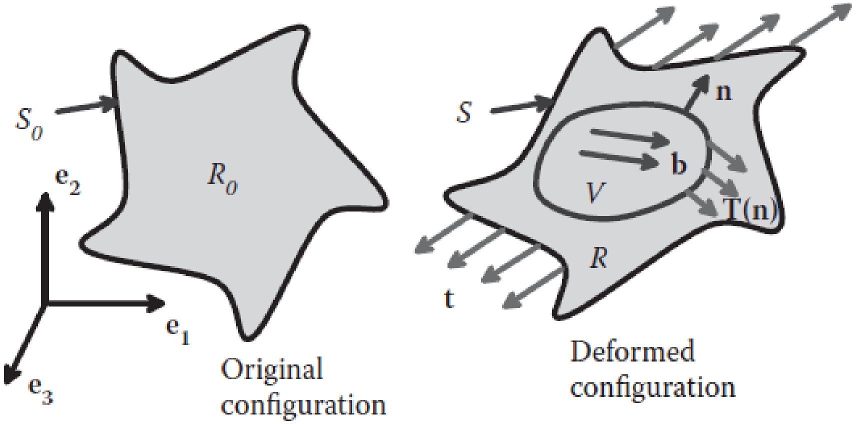
- Forces are vectors (unit: N)

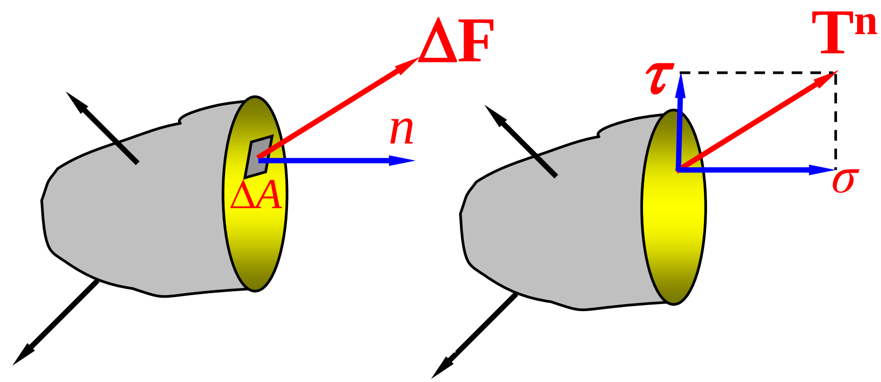
$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 = F_i\mathbf{e}_i$$

- Often interpreted in terms of density: body force density and surface force density

$$\boldsymbol{F}_R = \iiint_V \boldsymbol{F}(\boldsymbol{x})dV \qquad \boldsymbol{F}_S = \iint_S \boldsymbol{T}^n(\boldsymbol{x})dS$$

Traction/Stress Vector





- Given ΔF as the force transmitted across ΔA, a stress traction vector can be defined as

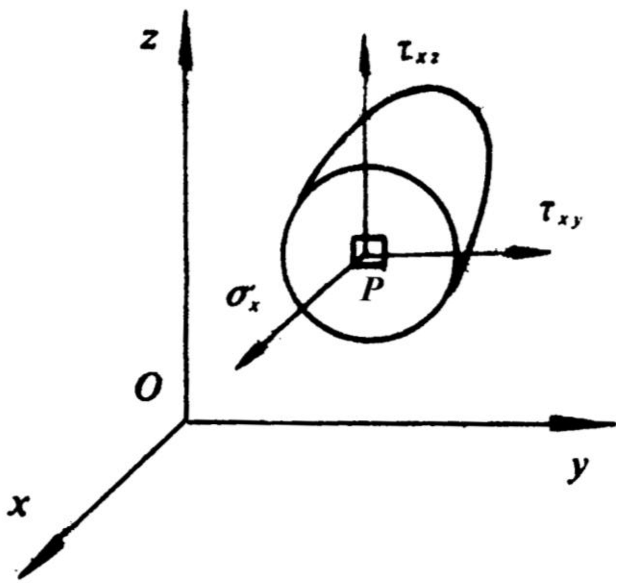
$$\mathbf{T}^{(\mathbf{n})}(\mathbf{x},\mathbf{n}) = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

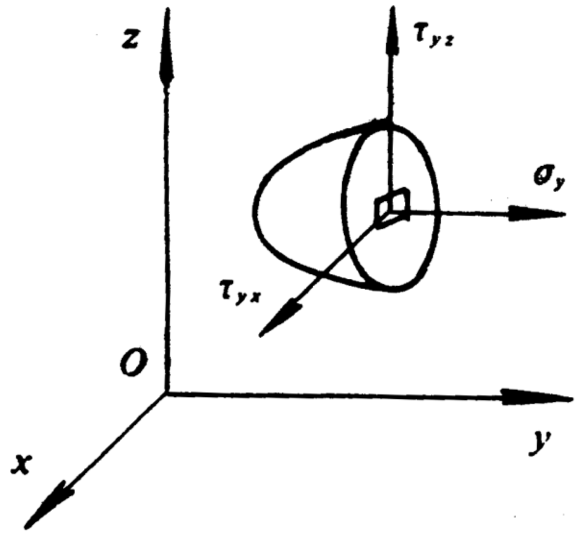
Units: Pa (N/m^2), 1 MPa = 10^6 Pa, 1 GPa = 10^9 Pa.

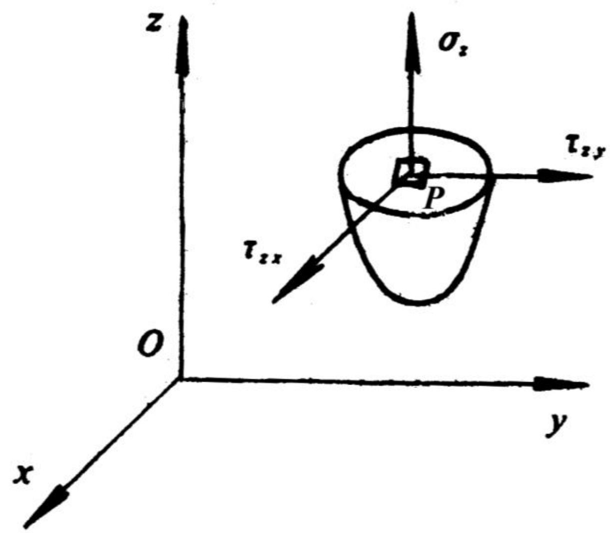
- Decomposition of the traction vector

$$\mathbf{T}^{(\mathbf{n})}(\mathbf{x},\mathbf{n}) = \sigma\,\mathbf{n} + \tau\mathbf{t} = \sigma\,\mathbf{n} + \tau'\mathbf{t}' + \tau''\mathbf{t}''$$

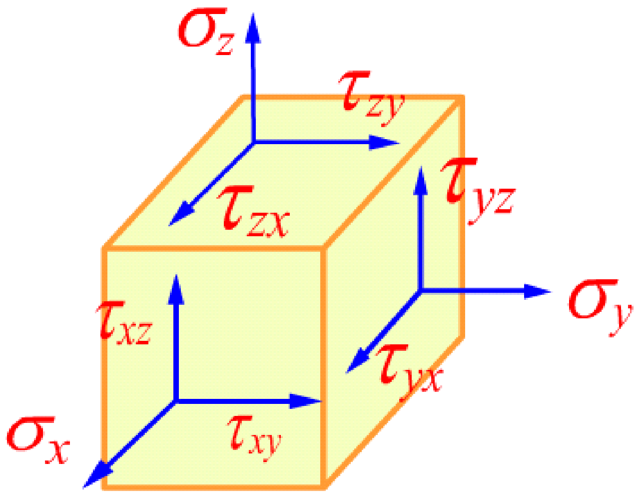
Cauchy Stress Tensor

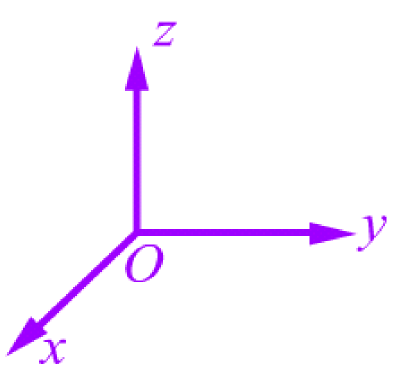






$\mathbf{T}^{(\mathbf{n})}(\mathbf{x}, \mathbf{n} = \mathbf{e}_x) = \sigma_x \mathbf{e}_x + \tau_{xy} \mathbf{e}_y + \tau_{xz} \mathbf{e}_z$	$\mathbf{T}^{(\mathbf{n})}(\mathbf{x}, \mathbf{n} = \mathbf{e}_y) = \tau_{yx} \mathbf{e}_x + \sigma_y \mathbf{e}_y + \tau_{yz} \mathbf{e}_z$	$\mathbf{T}^{(\mathbf{n})}(\mathbf{x}, \mathbf{n} = \mathbf{e}_z) = \tau_{zx} \mathbf{e}_x + \tau_{zy} \mathbf{e}_y + \sigma_z \mathbf{e}_z$



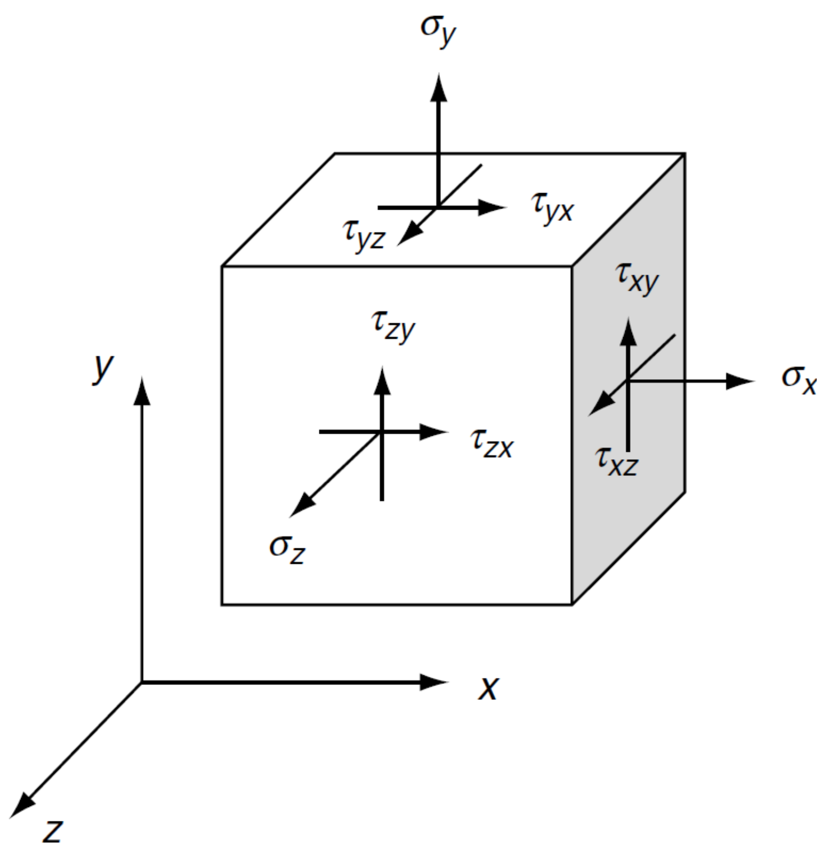


$$[\sigma_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

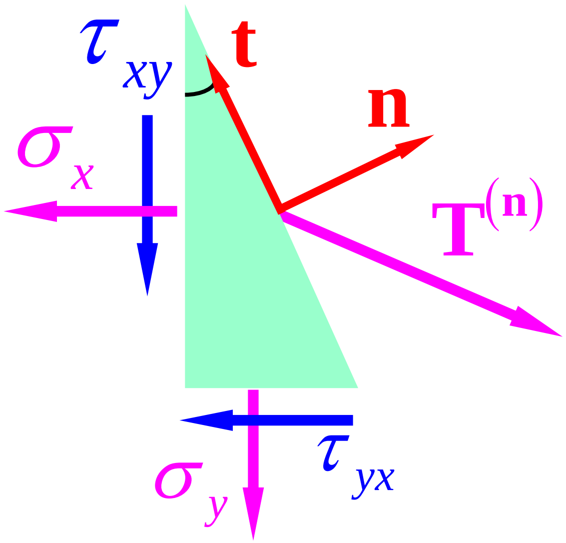
von Karman Notation

Sign Convention

- Normal stress: tension positive / compression negative
- Shear stress: product of the surface normal (the first subscript) and the stress direction (the second subscript)
- All stress components shown in the figure are positive.



Traction on an Oblique Plane - 2D



$$\begin{cases} 0 = \sum F_x \\ 0 = \sum F_y \end{cases}$$

$$\Rightarrow \begin{cases} T_x^{(\mathbf{n})} \Delta A = \sigma_x \Delta A \cos\theta + \tau_{yx} \Delta A \sin\theta \\ T_y^{(\mathbf{n})} \Delta A = \tau_{xy} \Delta A \cos\theta + \sigma_y \Delta A \sin\theta \end{cases}$$

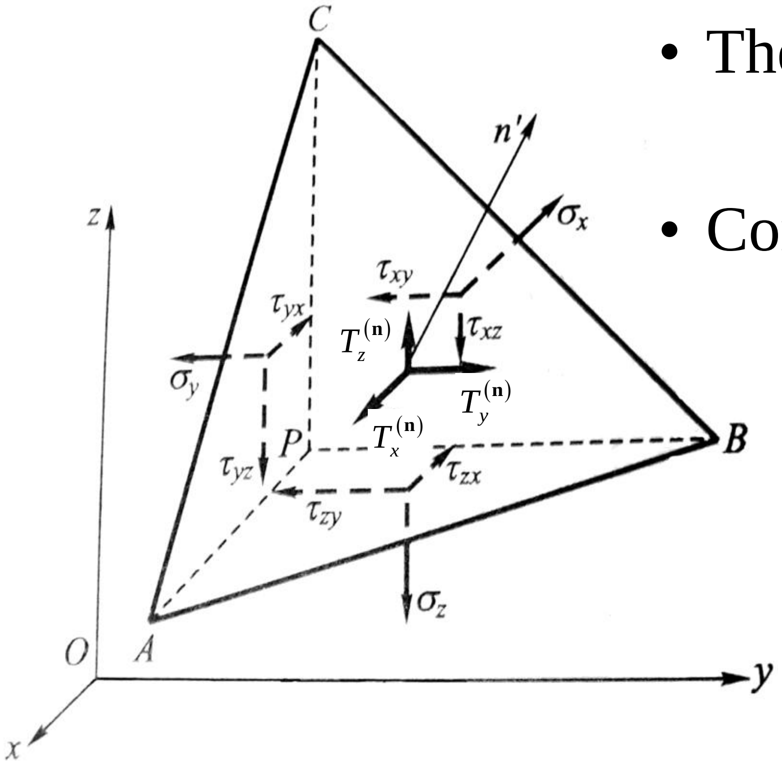
$$\Rightarrow \begin{cases} T_x^{(\mathbf{n})} = \sigma_x n_x + \tau_{yx} n_y \\ T_y^{(\mathbf{n})} = \tau_{xy} n_x + \sigma_y n_y \end{cases}$$

$$\Rightarrow \left\{ T_x^{(\mathbf{n})} \quad T_y^{(\mathbf{n})} \right\} = \left\{ n_x \quad n_y \right\} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} T_\alpha^{(\mathbf{n})} &= n_\beta \sigma_{\beta\alpha} \\ \mathbf{T}^{(\mathbf{n})} &= \mathbf{n} \cdot \boldsymbol{\sigma} \end{aligned}}$$

2D Cauchy's relation

Traction on an Oblique Plane - 3D



- The state of stress at a point is defined by:

$$\sigma_x, \tau_{xy}, \tau_{xz}, \tau_{yx}, \sigma_y, \tau_{yz}, \tau_{zx}, \tau_{zy}, \sigma_z$$

- Consider the tetrahedron with unit normal **n**

$$n_i = \frac{\mathbf{n}\cdot\mathbf{e}_i}{\|\mathbf{n}\|\,\|\mathbf{e}_i\|} = \cos\left(\mathbf{n}, \mathbf{e}_i\right)$$

$$\begin{cases} 0 = \sum F_x \\ 0 = \sum F_y \end{cases}$$

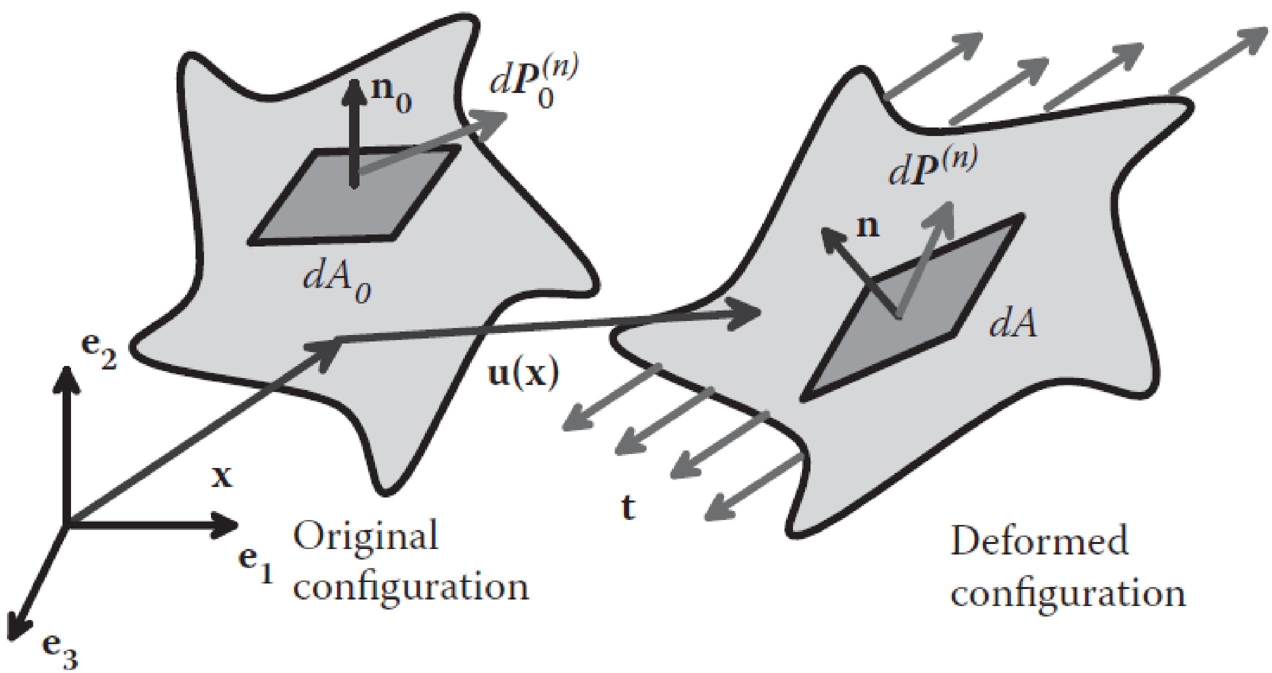
$$\Rightarrow \begin{cases} T_x^{(\mathbf{n})}\Delta A = \sigma_x \Delta A n_x + \tau_{yx}\Delta A n_y + \tau_{zx}\Delta A n_z \\ T_y^{(\mathbf{n})}\Delta A = \tau_{xy}\Delta A n_x + \sigma_y \Delta A n_y + \tau_{zy}\Delta A n_z \\ T_z^{(\mathbf{n})}\Delta A = \tau_{xz}\Delta A n_z + \tau_{yz}\Delta A n_y + \sigma_z \Delta A n_z \end{cases}$$

3D Cauchy's relation

$$\Rightarrow \boxed{\begin{array}{l} T_i^{(\mathbf{n})} = n_j \sigma_{ji} \\ \mathbf{T}^{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\sigma} \end{array}}$$

Different Stress Measures

- **Cauchy stress $\boldsymbol{\sigma_{ij}}$** (the actual force per unit area acting on an actual, deformed solid) is the most physical measure of internal force.
- Other definitions of stress often appear in constitutive equations.
- Other stress measures regard forces as acting on the undeformed solid.



Different Stress Measures

- **Cauchy stress σ_{ij}**
- **Kirchhoff stress** has no obvious physical significance.

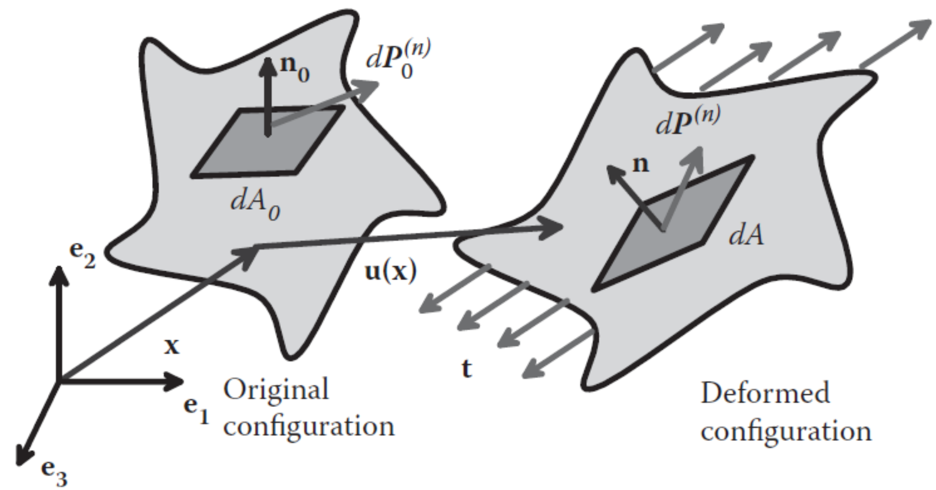
$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \qquad \tau_{ij} = J\sigma_{ij}$$

- **Nominal (1st Piola-Kirchhoff) stress / PK1 stress: S_{ij}**
- **Material (2nd Piola-Kirchhoff) stress / PK2 stress: Σ_{ij}**

$$dP_i^{(\mathbf{n})} = dA n_k \sigma_{ki}$$

$$dP_i^{(\mathbf{n})} = dA_0 n_k^0 S_{ki}$$

$$dP_i^{(\mathbf{n}0)} = dA_0 n_k^0 \Sigma_{ki}$$

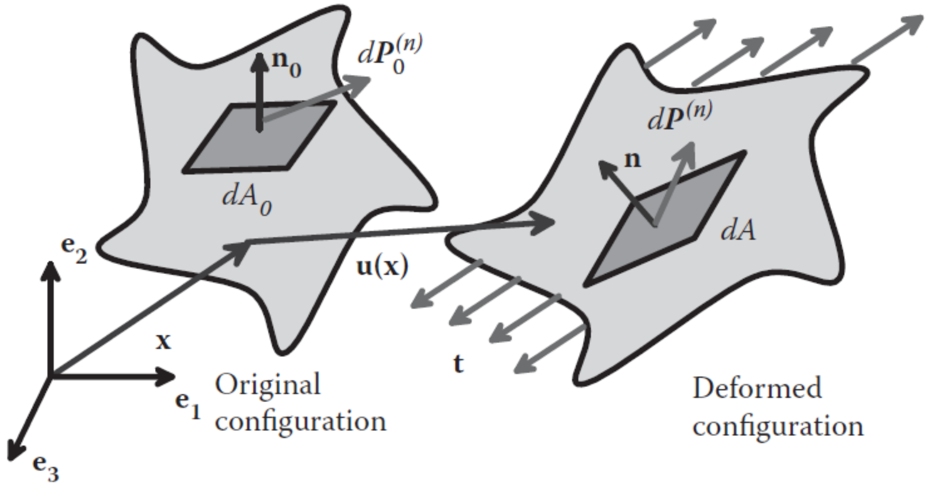


PK1 Stress

- **Nominal (first Piola-Kirchhoff) stress / PK1 stress:** internal force per unit **undeformed** area acting within the **deformed** solid.

$$\left.\begin{aligned} dAn_j &= JdA_0 n_k^0 F_{kj}^{-1} \\ dP_i^{(\mathbf{n})} &= \underline{dAn_j}\sigma_{ji} \end{aligned}\right\} \Rightarrow dP_i^{(\mathbf{n})} = \underline{J}dA_0 n_k^0 \underline{F_{kj}^{-1}\sigma_{ji}} = dA_0 n_k^0 S_{ki} \Rightarrow$$

$$\boxed{\begin{aligned} &S_{ki} = JF_{kj}^{-1}\sigma_{ji}, \quad \mathbf{S} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma} \\ &\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\mathbf{S} \quad \sigma_{ij} = \frac{1}{J}F_{ik}S_{kj} \end{aligned}}$$



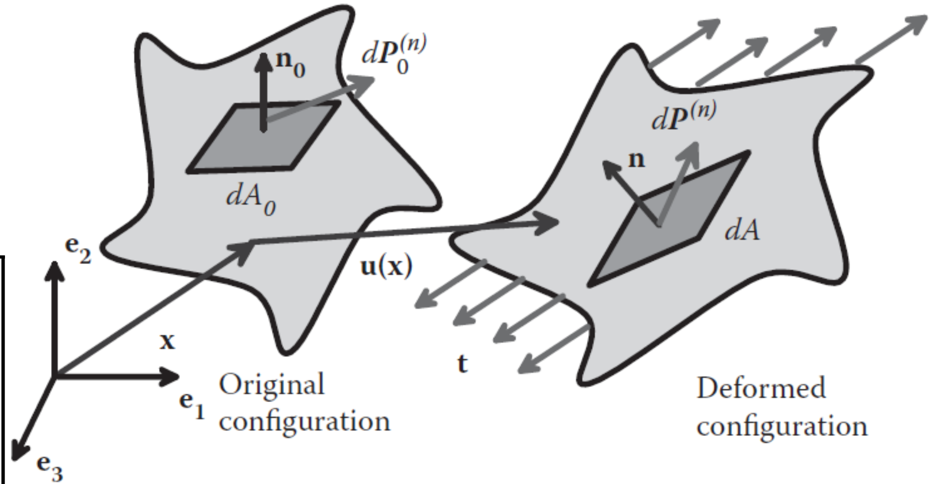
PK2 Stress

- **Material (second Piola-Kirchhoff) stress / PK2 stress:** force per unit **undeformed** area, as acting within the **undeformed** solid
- The infinitesimal force $d\mathbf{P}^{(\mathbf{n})}$ is assumed to behave like an infinitesimal material fiber in the solid

$$\left.\begin{aligned} dP_i^{(\mathbf{n}0)} &= dA_0 n_k^0 \Sigma_{ki} \\ dP_j^{(\mathbf{n})} &= dA_0 n_k^0 S_{kj} \end{aligned}\right\} \Rightarrow dP_i^{(\mathbf{n}0)} = F_{ij}^{-1} \underline{dP_j^{(\mathbf{n})}} = \underline{F_{ij}^{-1}} dA_0 n_k^0 \underline{S_{kj}} \Rightarrow$$

$$\boxed{\begin{aligned} &\Sigma_{ki} = S_{kj} F_{ij}^{-1} = J F_{kl}^{-1} \sigma_{lj} F_{ij}^{-1}, \\ &\mathbf{\Sigma} = \mathbf{S} \cdot \mathbf{F}^{-T} = J \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \end{aligned}}$$

$$\boxed{\begin{aligned} &S_{kl} = \Sigma_{ki} F_{li}, \sigma_{mn} = \frac{1}{J} F_{mk} \Sigma_{ki} F_{ni}, \\ &\mathbf{S} = \mathbf{\Sigma} \cdot \mathbf{F}^T, \boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \mathbf{\Sigma} \cdot \mathbf{F}^T \end{aligned}}$$



Different Stress Measures

- It is best not to try to attach too much physical significance to these stress measures.
- Cauchy stress is the best physical measure of internal force: it is the force per unit area acting inside the deformed solid.
- Other stress measures are best regarded as generalized forces which are work conjugate to particular strain measures. It means that the stress measure multiplied by the time derivative of the strain measure tells you the rate of work done by the forces.

$$\dot{W} = D_{ij}\sigma_{ij}dV = D_{ij}\tau_{ji}dV_0 = \dot{F}_{ij}S_{ji}dV_0 = \dot{E}_{ij}\Sigma_{ji}dV_0$$

Stress Measures for Infinitesimal Deformations

- For a problem involving infinitesimal deformation, shape changes are characterized by the infinitesimal strain tensor and rotation tensor.
- All the stress measures defined in the preceding section are approximately equal: $\sigma_{ij} \approx \tau_{ij} \approx S_{ij} \approx \Sigma_{ij}$

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}, J = \det(F) \approx 1 + \frac{\partial u_k}{\partial x_k}, \partial u_i / \partial x_j << 1$$

$$\sigma_{ij} = \frac{1}{J} F_{ik} S_{kj} = \frac{1}{(1 + \partial u_k / \partial x_k)} \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k} \right) S_{kj} \approx S_{ij}$$

$$\sigma_{mn} = \frac{1}{J} F_{mk} \Sigma_{ki} F_{ni} = \frac{1}{(1 + \partial u_k / \partial x_k)} \left(\delta_{mk} + \frac{\partial u_m}{\partial x_k} \right) \Sigma_{ki} \left(\delta_{ni} + \frac{\partial u_n}{\partial x_i} \right) \approx \Sigma_{mn}$$

Principal Stresses and Directions

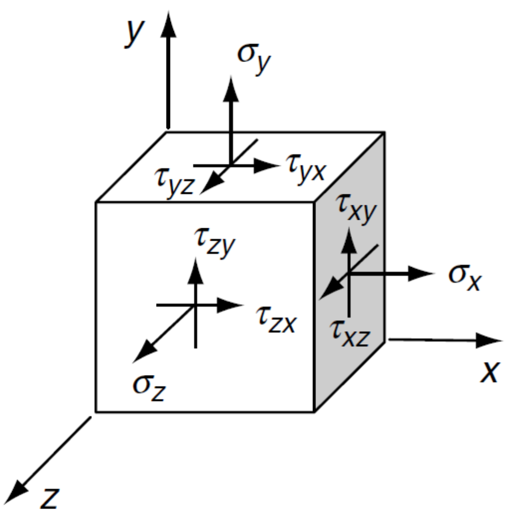
- Seeking the solution through an Eigen-equation

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\mathbf{n} = \sigma_n \mathbf{n} \Rightarrow \det\left[\sigma_{ij} - \sigma_n \delta_{ij}\right] = 0$$

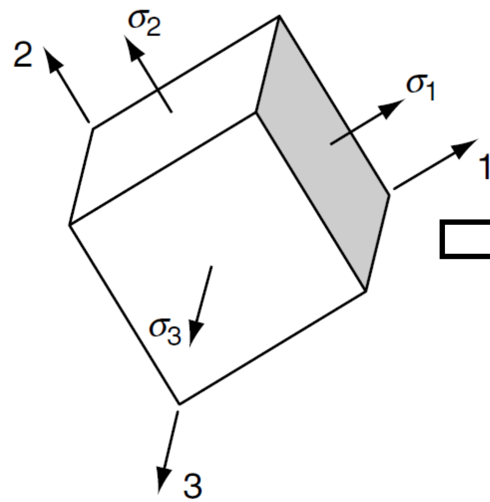
$$\Rightarrow \boxed{-\sigma_n{}^3 + I_1\sigma_n{}^2 - I_2\sigma_n + I_3 = 0}$$

- Three invariants of the stress tensor

$$I_1 = \sigma_{kk}, \quad I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right), \quad I_3 = \det\left[\sigma_{ij}\right].$$



(General Coordinate System)



(Principal Coordinate System)

$\Rightarrow$

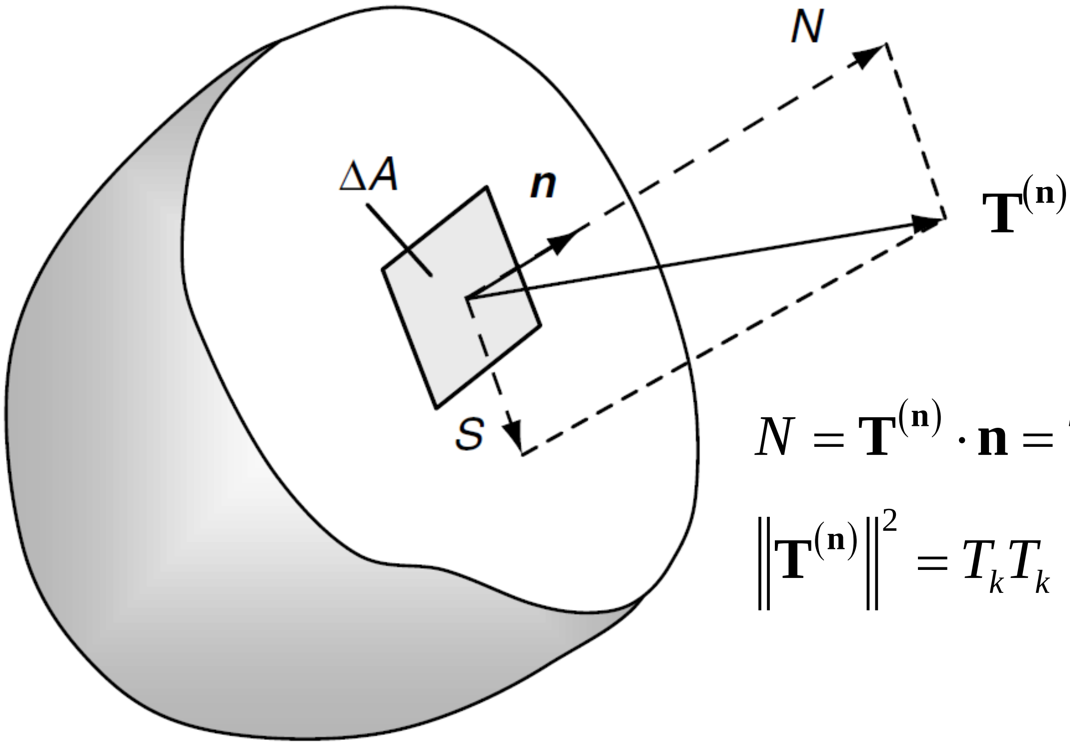
$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3,$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1,$$

$$I_3 = \sigma_1\sigma_2\sigma_3.$$

Traction Vector Decomposition



$$N^2 + S^2 = \left\| \mathbf{T}^{(\mathbf{n})} \right\|^2$$

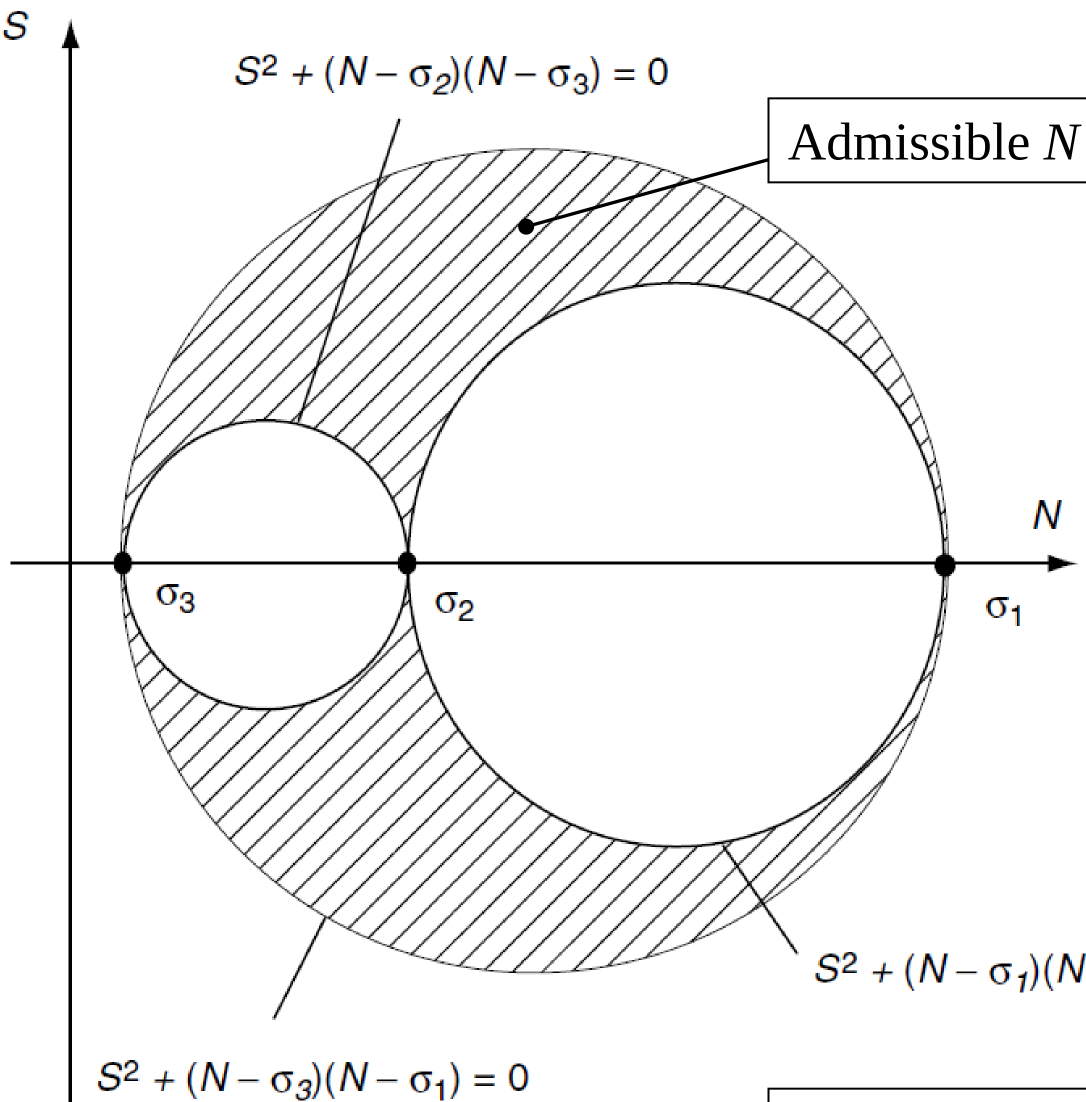
$$N = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = T_i n_i = \sigma_{ji} n_j n_i = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 = T_k T_k = \sigma_{ik} n_i \sigma_{jk} n_j = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

$$\Rightarrow \begin{cases} \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = N \\ \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = N^2 + S^2 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{cases} \Rightarrow \begin{cases} n_1^2 = \dfrac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\ n_2^2 = \dfrac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\ n_3^2 = \dfrac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \end{cases}$$

The principal space is taken as the reference.

Mohr's Circles of Stress



Admissible *N* and *S* values lie in the shaded area.

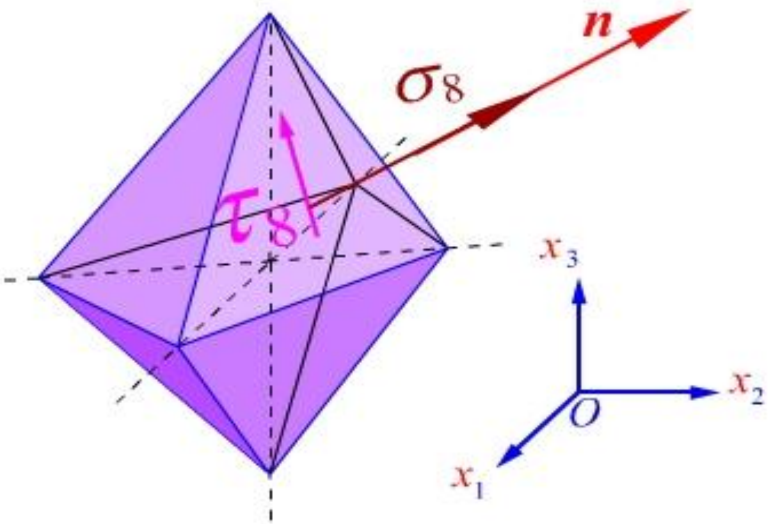
For $\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$\Rightarrow \begin{cases} S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0 \\ S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \\ S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0 \end{cases}$$

$$S_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

The principal space is taken as the reference.

Octahedral Stress



$$\mathbf{n} = \left(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3\right) / \sqrt{3}$$

$$\Rightarrow \begin{cases} \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = N \\ \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = N^2 + S^2 \end{cases}$$

$$\Rightarrow \boxed{N = \sigma_8 = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{1}{3} I_1}$$

$$\Rightarrow \boxed{\begin{aligned} S = \tau_8 &= \sqrt{\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - N^2} \\ &= \frac{1}{3}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2} \\ &= \frac{1}{3}\sqrt{\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 - 6\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)} \\ &= \frac{1}{3}\sqrt{2I_1^2 - 6I_2} \end{aligned}}$$

- Octahedral shear stress is the equivalent stress of the maximum distortion energy criterion.
- Extremely significant for plastic deformation

Sample Problem

- For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with the given unit normal. Also, determine the normal and shear stresses on this plane.

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{n} = \left\{ 0 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \right\}.$$

- Solution:

$$I_1 = 3, \quad I_2 = -6, \quad I_3 = -8$$

$$\Rightarrow -\sigma_n^{\ 3} + 3\sigma_n^{\ 2} + 6\sigma_n - 8 = 0$$

$$\Rightarrow \sigma_1 = 4, \quad \sigma_2 = 1, \quad \sigma_3 = -2.$$

$$\Rightarrow \begin{cases} -n_1^{(1)} + n_2^{(1)} + n_3^{(1)} = 0 \\ n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} = 0 \\ n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} = 0 \end{cases}$$

$$\Rightarrow \mathbf{n}^{(1)} = \left(2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right) / \sqrt{6}$$

$$\Rightarrow \begin{cases} \mathbf{n}^{(2)} = \left(-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right) / \sqrt{3} \\ \mathbf{n}^{(3)} = \left(-\mathbf{e}_2 + \mathbf{e}_3\right) / \sqrt{2} \end{cases}$$

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} = \left\{ 0 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \right\} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \sqrt{2}\left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right)$$

$$N = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = \sqrt{2}\left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right) \cdot \left(\mathbf{e}_2 + \mathbf{e}_3\right) / \sqrt{2} = 2$$

$$S = \sqrt{\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 - N^2} = \sqrt{2}$$

Hydrostatic, Deviatoric and von Mises Stress

- Hydrostatic (mean) stress tensor: only measures volume change

$$\tilde{\sigma}_{ij} = \sigma_h \delta_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)\delta_{ij}$$

- Deviatoric (octahedral) stress tensor: shape change $\sigma'_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij}$
- The von Mises effective stress can be regarded as a uniaxial equivalent of a multi-axial stress state:

$$\sigma_e \equiv \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sqrt{\frac{1}{2}\left\{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right\}}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \sigma_h = \frac{1}{3}\sigma, \boldsymbol{\sigma}' = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix} \Rightarrow \sigma_e \equiv \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma$$

- The hydrostatic and von Mises stresses are invariants of the stress tensor: they have the same value regardless of the basis.

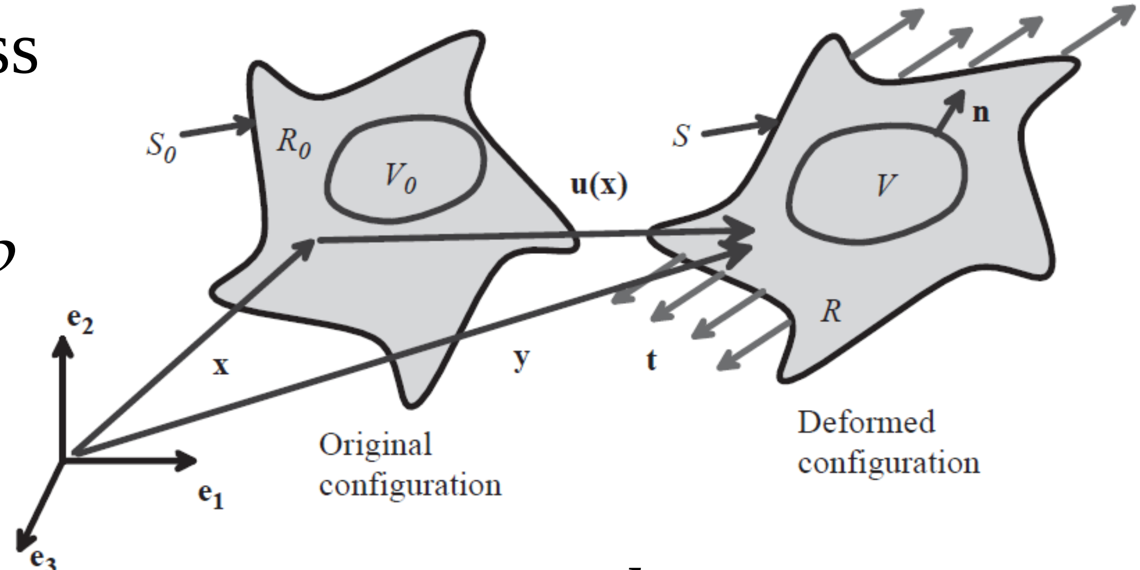
Conservation of Mass

- Conservation of mass

$$\rho dV = \rho_0 dV_0$$

$$\Rightarrow J = dV / dV_0 = \rho_0 / \rho$$

- Time derivative of linear momentum



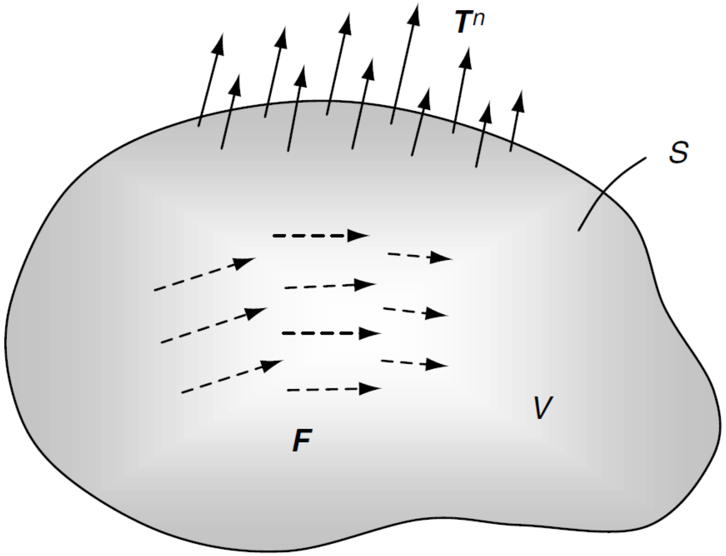
$$\frac{d}{dt}\iiint_V \rho v_i dV = \frac{d}{dt}\iiint_{V_0} \rho_0 v_i dV_0 = \iiint_{V_0} \rho_0 \frac{dv_i}{dt} dV_0$$

- Time derivative of angular momentum

$$\frac{d}{dt}\iiint_V \varepsilon_{ijk} y_j v_k \underline{\rho dV} = \frac{d}{dt}\iiint_{V_0} \varepsilon_{ijk} \underline{y_j v_k} \rho_0 dV_0 = \iiint_{V_0} \varepsilon_{ijk} \left(\frac{dy_j}{dt} v_k + y_j \frac{dv_k}{dt} \right) \rho_0 dV_0$$

$$= \iiint_{V_0} \left(\cancel{\varepsilon_{ijk} v_j v_k} + \varepsilon_{ijk} y_j \frac{dv_k}{dt} \right) \underline{\rho_0 dV_0} = \iiint_V \varepsilon_{ijk} y_j \frac{dv_k}{dt} \rho dV$$

Conservation of Linear Momentum



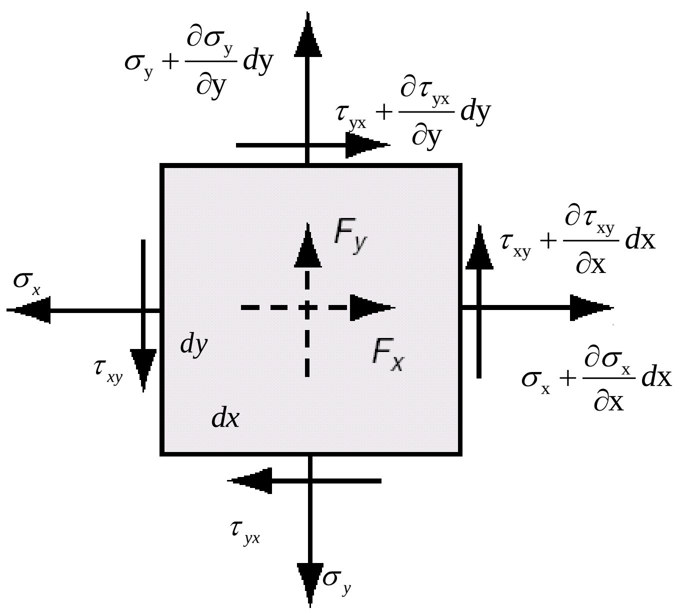
$$\iiint_V \rho \frac{dv_i}{dt} dV = \iint_S T_i^n dS + \iiint_V F_i dV = \iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV$$

$$= \iiint_V \left(\sigma_{ji,j} + F_i \right) dV$$

$$\Rightarrow \boxed{\sigma_{ji,j} + F_i = \rho a_i} \Rightarrow$$

3-D

$$\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \tau_{21}}{\partial y_2} + \frac{\partial \tau_{31}}{\partial y_3} + F_1 = \rho a_1 \\
&\frac{\partial \tau_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \tau_{23}}{\partial y_3} + F_2 = \rho a_2 \\
&\frac{\partial \tau_{13}}{\partial y_1} + \frac{\partial \tau_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + F_3 = \rho a_3
\end{aligned}$$



2-D

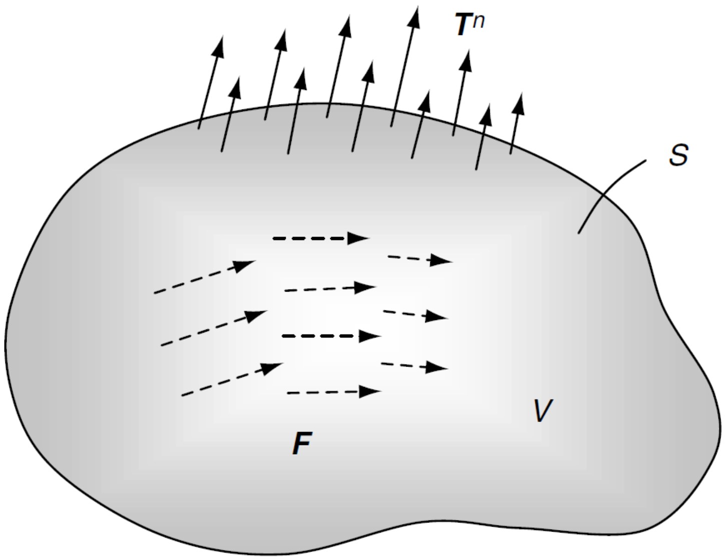
$$y_1 \equiv x, y_2 \equiv y \Rightarrow$$

$$\rho a_x = \sum F_x = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dy - \sigma_x dy$$

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx - \tau_{yx} dx + F_x dxdy$$

$$\Rightarrow \boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = \rho a_x}$$

Conservation of Angular Momentum



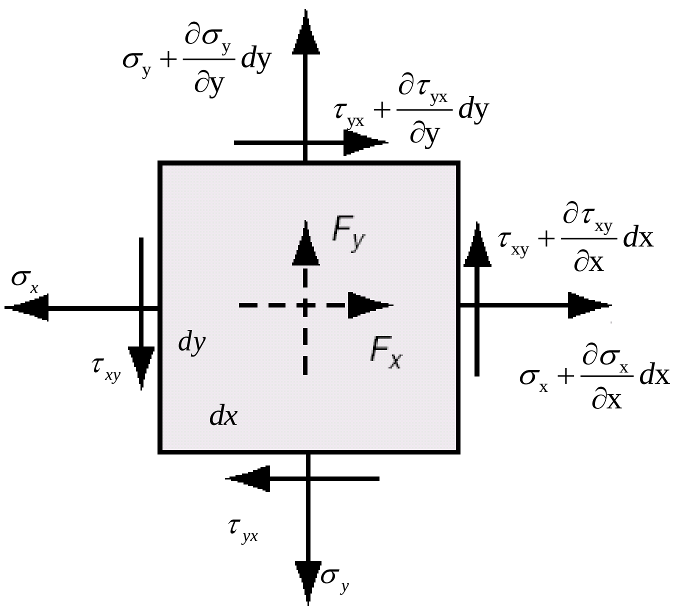
$$\frac{d}{dt}\iiint_V \mathbf{r}\times\rho\mathbf{v}\,dV = \iint_S \mathbf{r}\times\mathbf{T^n}\,dS + \iiint_V \mathbf{r}\times\mathbf{F}\,dV$$

$$= \iint_S \varepsilon_{ijk} y_j \sigma_{lk} n_l \,dS + \iiint_V \varepsilon_{ijk} y_j F_k \,dV$$

$$= \iiint_V \left(\left(\varepsilon_{ijk} y_j \sigma_{lk}\right)_{,l} + \varepsilon_{ijk} y_j F_k\right) dV$$

$$\Rightarrow 0 = \iiint_V \left(\varepsilon_{ijk}\delta_{jl}\sigma_{lk} + \cancel{\varepsilon_{ijk} y_j \left(\sigma_{lk,l} + F_k - \rho a_k\right)}\right) dV$$

$$= \iiint_V \varepsilon_{ijk}\sigma_{jk}\,dV \quad \Rightarrow \boxed{\sigma_{jk} = \sigma_{kj} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T}$$



2-D

$$y_1 \equiv x, y_2 \equiv y \Rightarrow$$

$$0 = \sum M = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy \frac{1}{2} dx + \tau_{xy} dy \frac{1}{2} dx$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx \frac{1}{2} dy - \tau_{yx} dx \frac{1}{2} dy$$

$$\Rightarrow \boxed{\tau_{xy} = \tau_{yx}}$$

Conservation Laws in terms of PK1 & PK2

- Conservation of linear momentum

$$\left.\begin{aligned} n_j dS &= J dS_0 n_k^0 F_{kj}^{-1} \\ J &= dV/dV_0 = \rho_0/\rho \\ S_{ki} &= J F_{kj}^{-1} \sigma_{ji} \end{aligned}\right\} \quad \Rightarrow \iiint_V a_i \underline{\rho dV} = \iint_S \sigma_{ji} \underline{n_j dS} + \iiint_V F_i \underline{dV}$$

$$\Rightarrow \iiint_{V_0} \rho_0 a_i dV_0 = \iint_{S_0} \underline{J F_{kj}^{-1} \sigma_{ji}} n_k^0 dS_0 + \iiint_{V_0} F_i \underline{J} dV_0 \quad \Rightarrow \iiint_{V_0} \left(S_{ki,k} + \frac{\rho_0}{\rho} F_i - \rho_0 a_i \right) dV_0 = 0 \Rightarrow$$

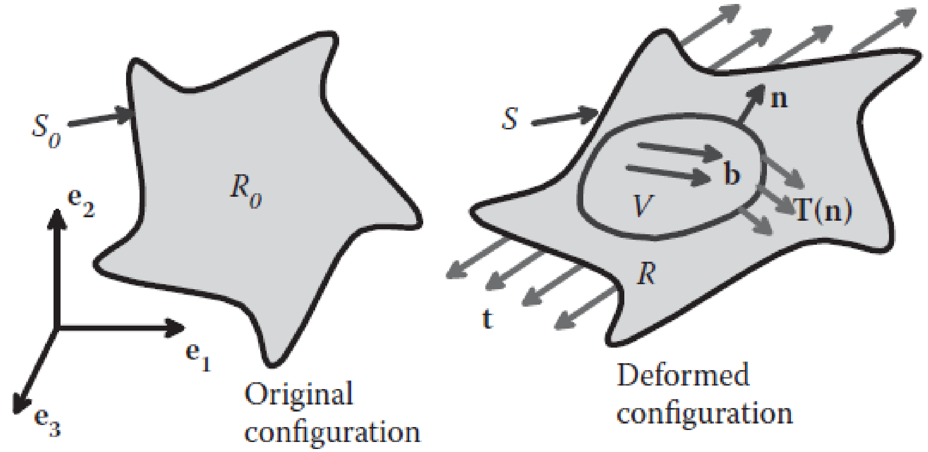
$$\boxed{\frac{\partial S_{ki}}{\partial x_k} + \frac{\rho_0}{\rho} F_i = \rho_0 a_i, \nabla \cdot \mathbf{S} + \frac{\rho_0}{\rho} \mathbf{F} = \rho_0 \mathbf{a}} \Rightarrow \boxed{\frac{\partial \left(\Sigma_{kj} F_{ij} \right)}{\partial x_k} + \frac{\rho_0}{\rho} F_i = \rho_0 a_i, \nabla \cdot \left(\mathbf{\Sigma} \cdot \mathbf{F}^T \right) + \frac{\rho_0}{\rho} \mathbf{F} = \rho_0 \mathbf{a}}$$

- Conservation of angular momentum

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \mathbf{S} = \frac{1}{J} \mathbf{F} \cdot \mathbf{\Sigma} \cdot \mathbf{F}^T, \quad \boldsymbol{\sigma}^T = \frac{1}{J} \mathbf{S}^T \cdot \mathbf{F}^T = \frac{1}{J} \mathbf{F} \cdot \mathbf{\Sigma}^T \cdot \mathbf{F}^T$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad \Rightarrow \boxed{\mathbf{F} \cdot \mathbf{S} = \left(\mathbf{F} \cdot \mathbf{S} \right)^T = \mathbf{S}^T \cdot \mathbf{F}^T, \quad \mathbf{\Sigma} = \mathbf{\Sigma}^T}$$

Rate of Work Done by Cauchy Stress



$$\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV$$

$$= \iint_S n_j \sigma_{ji} v_i dS + \iiint_V F_i v_i dV$$

$$= \iiint_V \left\{ \frac{\partial}{\partial y_j}\left(\sigma_{ji} v_i\right) + F_i v_i \right\} dV$$

$$= \iiint_V \left\{ \sigma_{ji} \frac{\partial v_i}{\partial y_j} + \left(\frac{\partial \sigma_{ji}}{\partial y_j} + F_i \right) v_i \right\} dV$$

$$= \iiint_V \left\{ \sigma_{ji}\left(D_{ij} + W_{ij}\right) + \rho a_i v_i \right\} dV = \iiint_V \sigma_{ji} D_{ij} dV + \iiint_V \rho a_i v_i dV$$

$$= \iiint_V \sigma_{ji} D_{ij} dV + \iiint_{V_0} \rho_0 \frac{1}{2} \frac{d}{dt}\left(v_i v_i\right) dV_0 = \iiint_V \sigma_{ji} D_{ij} dV + \frac{d}{dt} \iiint_V \frac{1}{2} \rho v_i v_i dV$$

$$\boxed{\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iiint_V \sigma_{ij} D_{ij} dV + \frac{d}{dt} \iiint_V \frac{1}{2} \rho v_i v_i dV}$$

- **Rate of work done by Kirchhoff stress**

$$\dot{W} = \iiint_{V_0} \sigma_{ij} D_{ij} J dV_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \quad \Rightarrow \boxed{\dot{W} = \iiint_{V_0} \tau_{ij} D_{ij} dV_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0}$$

Rate of Work Done by PK1 & PK2 Stresses

$$T_i^n dS = n_j^0 S_{ji} dS_0, dV = JdV_0 \Rightarrow \dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iint_S n_j^0 S_{ji} v_i dS_0 + \iiint_{V_0} F_i v_i J dV_0$$

$$= \iiint_{V_0} \left\{ \frac{\partial}{\partial x_j}\left(S_{ji} v_i\right) + JF_i v_i \right\} dV_0 = \iiint_{V_0} \left\{ S_{ji} \frac{\partial v_i}{\partial x_j} + \left(\frac{\partial S_{ji}}{\partial x_j} + JF_i \right) v_i \right\} dV_0 = \iiint_{V_0} S_{ji} \dot{F}_{ij} dV_0 + \iiint_{V_0} \rho_0 a_i v_i dV_0$$

$$\Rightarrow \boxed{\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iiint_{V_0} S_{ij} \dot{F}_{ji} dV_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0}$$

$$S_{ij} \dot{F}_{ji} = \left(\Sigma_{ik} F_{jk}\right) \dot{F}_{ji} = \frac{1}{2}\left(\Sigma_{ik} + \Sigma_{ki}\right) F_{jk} \dot{F}_{ji} = \frac{1}{2}\left(\Sigma_{ik} \dot{F}_{ji} F_{jk} + \Sigma_{ki} F_{jk} \dot{F}_{ji}\right)$$

$$= \frac{1}{2}\left(\Sigma_{ik} \dot{F}_{ji} F_{jk} + \Sigma_{ik} F_{ji} \dot{F}_{jk}\right) = \Sigma_{ik} \frac{1}{2}\left(\dot{F}_{ji} F_{jk} + F_{ji} \dot{F}_{jk}\right) = \Sigma_{ik} \dot{E}_{ik}$$

$$\Rightarrow \boxed{\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iiint_{V_0} \Sigma_{ik} \dot{E}_{ik} dV_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0}$$

$$E_{ik} = \frac{1}{2}\left(C_{ik} - \delta_{ik}\right) = \frac{1}{2}\left(F_{ji} F_{jk} - \delta_{ik}\right) \Rightarrow \dot{E}_{ik} = \frac{1}{2} \dot{C}_{ik} = \frac{1}{2}\left(\dot{F}_{ji} F_{jk} + F_{ji} \dot{F}_{jk}\right)$$

- **Rate of work done by infinitesimal stress**

$$\Rightarrow \boxed{\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iiint_{V_0} \sigma_{ik} \dot{\varepsilon}_{ik} dV_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0}$$