Simple Elastoplastic BVPs

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Outline

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Assumptions

- Body force density is given.
- Prescribed boundary tractions and/or displacements
- All displacements are small. This means that we can use the infinitesimal strain tensor to characterize deformation; we do not need to distinguish between stress measures, and we do not need to distinguish between deformed and undeformed configurations of the solid when writing equilibrium equations and boundary conditions.
- The material is isotropic, elastic-perfectly plastic solid.
- Neglect temperature changes.

Introduction

- **The elastic limit:** This is the load required to initiate plastic flow in the solid.
- **The plastic collapse load:** At this load, the displacements in the solid become infinite.
- **Residual stress:** If a solid is loaded beyond the elastic limit and then unloaded, a system of self-equilibrated stress is established in the material.
- **Shakedown**: If an elastic-plastic solid is subjected to cyclic loading and the maximum load during the cycle exceeds yield, then some plastic deformation must occur in the material during the first load cycle. However, residual stresses are introduced in the solid, which may prevent plastic flow during subsequent cycles of load. This process is known as "shakedown," and the maximum load for which it can occur is known as the shakedown limit. The shakedown limit is often substantially higher than the elastic limit, so the concept of shakedown can often be used to reduce the weight of a design.
- **Cyclic plasticity:** For cyclic loads exceeding the shakedown limit, a region in the solid will be repeatedly plastically deformed.

Summary of Governing Equations

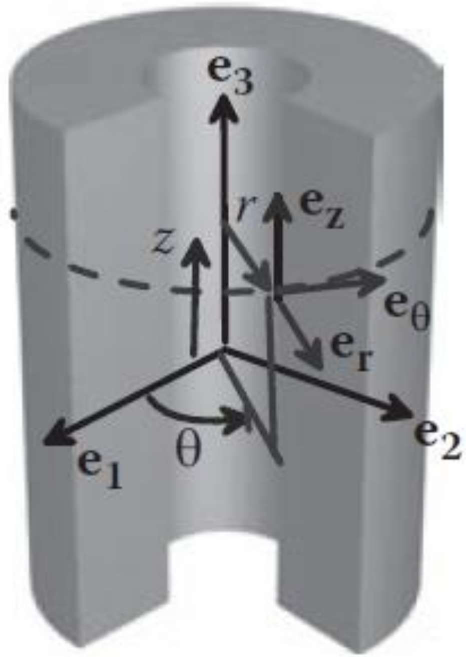
- Displacement-strain relation: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Strain partition: $d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$
- Incremental stress-stain relation:

$$d\varepsilon_{ij}^{e} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij}; \quad d\varepsilon_{ij}^{p} = \begin{cases} 0, & \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} < \sigma_Y \\ d\bar{\varepsilon}^{p}\frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}, & \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma_Y \end{cases}$$

- Equations of static equilibrium: $\sigma_{ji,j} + F_i = 0.$
- Traction BCs on S_t: $\sigma_{ij} n_i = t_j$
- Displacement BCs on S_u: $u_i = \bar{u}_i$

Cylindrically Symmetric Elastoplastic Solids

- Cylindrically symmetric geometry and loading (i.e. internal body forces, tractions or displacements BCs, nonuniform temperature distribution).
- Cylindrical-polar bases: $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$
- Cylindrical-polar coordinates: $\{r, \theta, z\}$
- Position vector: $\mathbf{x} = r\mathbf{e}_r$
- Displacement vector: $\mathbf{u} = u_r[r]\mathbf{e}_r$
- Body force vector: $\mathbf{F} = F_r[r]\mathbf{e}_r$
- Acceleration vector: $\mathbf{a} = -\omega^2 r\mathbf{e}_r$



Cylindrically Symmetric Elastoplastic Solids

- Cauchy stress: $\boldsymbol{\sigma} = \sigma_r[r]\mathbf{e}_r\mathbf{e}_r + \sigma_\theta[r]\mathbf{e}_\theta\mathbf{e}_\theta + \sigma_z[r]\mathbf{e}_z\mathbf{e}_z$
- Infinitesimal strain: $\boldsymbol{\varepsilon} = \varepsilon_r[r]\mathbf{e}_r\mathbf{e}_r + \varepsilon_\theta[r]\mathbf{e}_\theta\mathbf{e}_\theta + \varepsilon_z[r]\mathbf{e}_z\mathbf{e}_z$
- Strain-displacement relation: $\varepsilon_r = \dfrac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \dfrac{u_r}{r}$
- Stress-strain relation in elastic region (plane strain or generalized plane strain):

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\varepsilon_r + \nu\varepsilon_\theta + \nu\varepsilon_z\right\},$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\varepsilon_r + (1-\nu)\varepsilon_\theta + \nu\varepsilon_z\right\},$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\varepsilon_r + \nu\varepsilon_\theta + (1-\nu)\varepsilon_z\right\}.$$

- von Mises yield criterion:

$$\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2\right\}} = \sigma_Y$$

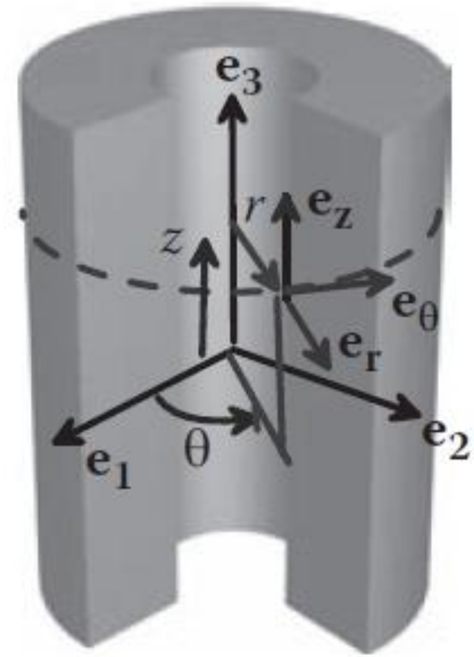
Cylindrically Symmetric Elastoplastic Solids

- **Stress-strain relation in plastic region**
- Strain partition: $d\varepsilon_r = d\varepsilon_r^e + d\varepsilon_r^p, d\varepsilon_\theta = d\varepsilon_\theta^e + d\varepsilon_\theta^p, d\varepsilon_z = d\varepsilon_z^e + d\varepsilon_z^p$
- Elastic strain: $d\varepsilon_r^e = \dfrac{d\sigma_r}{E} - \dfrac{\nu\left(d\sigma_\theta + d\sigma_z\right)}{E}, \cdots, \cdots$
- Flow rule:

$$d\varepsilon_r^p = d\bar{\varepsilon}^p \frac{3}{2}\frac{\sigma_r'}{\sigma_Y} = d\bar{\varepsilon}^p \frac{3}{2}\frac{1}{\sigma_Y}\left\{\sigma_r - \frac{1}{3}\left(\sigma_r + \sigma_\theta + \sigma_z\right)\right\} = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\sigma_r - \frac{1}{2}\left(\sigma_\theta + \sigma_z\right)\right\}, \cdots, \cdots$$

- Equations of motion: $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} + F_r = -\rho\omega^2 r$
- Traction BCs: $\sigma_R\left[a\right] = \sigma_a, \sigma_R\left[b\right] = \sigma_b.$
- BCs: $u_r\left[a\right] = u_a,\ \ u_r\left[b\right] = u_b;$ or $\sigma_r\left[a\right] = \sigma_a,\ \ \sigma_r\left[b\right] = \sigma_b$
- There is no clean, direct, and general method for integrating these equations. Instead, solutions must be found using a combination of physical intuition and some algebraic tricks.

Hollow Cylinder under Monotonic Pressure



- We consider a long hollow cylinder.
- The sphere is stress free before it is loaded.
- No body forces act on the cylinder.
- The cylinder has zero angular velocity.
- The cylinder has uniform temperature.
- The cylinder does not stretch parallel to its axis.
- The inner surface $r = a$ is subjected to monotonically increasing pressure p_a.
- The outer surface $r = b$ is traction free.
- Strains are infinitesimal.
- We aim to find….

Hollow Cylinder under Monotonic Pressure

- Elastic solution $u_r = \dfrac{(1+\nu)\,p_a a^2 r}{E\left(b^2 - a^2\right)}\left\{(1-2\nu) + \dfrac{b^2}{r^2}\right\};$

$$\sigma_r = \frac{p_a a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\}, \sigma_\theta = \frac{p_a a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right), \sigma_z = \frac{2\nu\, p_a a^2}{b^2 - a^2}.$$

- von Mises effective stress:

$$\sigma_e = \frac{p_a a^2}{b^2 - a^2}\sqrt{\frac{3b^4}{r^4} + 1 + 4\nu^2 - 4\nu}, \quad \nu \approx 0.5 \Rightarrow \quad \sigma_e \approx \frac{\sqrt{3}\,p_a a^2 b^2}{\left(b^2 - a^2\right) r^2}$$

- We see that the hollow cylinder first reaches yield at $r = a$, with the elastic limit: $p_a/\sigma_Y \approx \left(1 - a^2/b^2\right)/\sqrt{3}$
- If the pressure is increased beyond yield, we anticipate that a region $a < r < c$ will deform plastically, whereas a region $c < r < b$ remains elastic.

Hollow Cylinder under Monotonic Pressure

- **In the plastic region *a* < *r* < *c***
- To simplify the calculation, we assume: $d\varepsilon_z^e = 0, d\varepsilon_z^p = 0$
- This assumption turns out to be exact for ν = 0.5 but is approximate for other values of Poisson's ratio.
- The plastic flow rule shows that

$$0 = d\varepsilon_z^p = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\sigma_z - \frac{1}{2}(\sigma_r + \sigma_\theta)\right\} \quad \Rightarrow \boxed{\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta)}$$

- Yield criterion

$$\sigma_Y = \sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2\right\}} = \sqrt{\frac{3}{4}(\sigma_r - \sigma_\theta)^2}$$

$$\sigma_\theta > 0, \sigma_r < 0 \quad \Rightarrow \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r) = \sigma_Y \quad \Rightarrow \sigma_\theta - \sigma_r = \frac{2\sigma_Y}{\sqrt{3}}$$

Hollow Cylinder under Monotonic Pressure

- **In the plastic region *a* < *r* < *c***
- Equation of static equilibrium

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}+\cancel{F_r}=-\cancel{\rho\omega^2}r \quad \Rightarrow \frac{d\sigma_r}{dr}-\frac{2\sigma_Y}{\sqrt{3}r}=0 \quad \Rightarrow \sigma_r=\frac{2\sigma_Y}{\sqrt{3}}\ln r+D$$

- Integrate and apply the BCs at $R = a$

$$\sigma_r\left[a\right]=-p_a \quad \Rightarrow \sigma_r=\frac{2\sigma_Y}{\sqrt{3}}\ln r/a-p_a$$

$$\sigma_\theta-\sigma_r=\frac{2\sigma_Y}{\sqrt{3}}, \quad \Rightarrow \sigma_\theta=\frac{2\sigma_Y}{\sqrt{3}}\ln r/a-p_a+\frac{2\sigma_Y}{\sqrt{3}}$$

- Elastic strain:

$$d\varepsilon_r^e=\frac{d\sigma_r}{E}-\frac{\nu\left(d\sigma_\theta+d\sigma_z\right)}{E}, d\varepsilon_\theta^e=\frac{d\sigma_\theta}{E}-\frac{\nu\left(d\sigma_z+d\sigma_r\right)}{E}, d\varepsilon_z^e=0.$$

Hollow Cylinder under Monotonic Pressure

- **In the plastic region *a* < *r* < *c***
- The plastic strains satisfy:

$$d\varepsilon_r^p = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\sigma_r - \frac{1}{2}(\sigma_\theta + \sigma_z)\right\}, d\varepsilon_\theta^p = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\sigma_\theta - \frac{1}{2}(\sigma_z + \sigma_r)\right\}$$

$$\Rightarrow d\varepsilon_r^p + d\varepsilon_\theta^p = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\sigma_r + \sigma_\theta - \frac{1}{2}(\sigma_r + \sigma_\theta + 2\sigma_z)\right\} = \frac{d\bar{\varepsilon}^p}{\sigma_Y}\left\{\frac{1}{2}(\sigma_r + \sigma_\theta) - \sigma_z\right\} = 0.$$

- The elastic strains thus satisfy (plane strain condition):

$$d\varepsilon_r + d\varepsilon_\theta = d\varepsilon_r^e + d\varepsilon_\theta^e = \frac{(d\sigma_r + d\sigma_\theta)}{E} - \frac{\nu(d\sigma_r + d\sigma_\theta + 2d\sigma_z)}{E} = \frac{(1+\nu)(1-2\nu)}{E}(d\sigma_r + d\sigma_\theta)$$

- Since the pressure is monotonically increasing, the incremental stress-strain relation can be integrated

$$\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r} = \frac{1}{r}\frac{d}{dr}(ru_r) = \varepsilon_r + \varepsilon_\theta = \frac{(1+\nu)(1-2\nu)}{E}(d\sigma_r + d\sigma_\theta) = \frac{(1+\nu)(1-2\nu)}{E}\left(\frac{4\sigma_Y}{\sqrt{3}}\ln r/a - 2p_a + \frac{2\sigma_Y}{\sqrt{3}}\right)$$

$$\Rightarrow u_r = \frac{(1+\nu)(1-2\nu)r}{E}\left(\frac{2\sigma_Y}{\sqrt{3}}\ln r/a - p_a\right) + \frac{C}{r}$$

Hollow Cylinder under Monotonic Pressure

- **In the elastic region** $\boldsymbol{c < r < b}$

$$\sigma_r = \frac{p_c c^2}{b^2 - c^2}\left\{1 - \frac{b^2}{r^2}\right\}, \sigma_\theta = \frac{p_c c^2}{b^2 - c^2}\left(1 + \frac{b^2}{r^2}\right), \sigma_z = \frac{2\nu p_c c^2}{b^2 - c^2}; u_r = \frac{(1+\nu) p_c c^2 r}{E\left(b^2 - c^2\right)}\left\{(1-2\nu) + \frac{b^2}{r^2}\right\}$$

- Form the radial stress in the plastic region, we obtain the pressure at the elastic-plastic boundary $r = c$

$$\sigma_r[c] = \frac{2\sigma_Y}{\sqrt{3}} \ln c/a - p_a = -p_c \quad \Rightarrow p_c = p_a - \frac{2\sigma_Y}{\sqrt{3}} \ln c/a$$

- The elastic-plastic boundary is located by noting that the stress in the elastic region must just reach yield at $r = c$.

$$\frac{2\sigma_Y}{\sqrt{3}} = \sigma_\theta[c] - \sigma_r[c] = \frac{p_c c^2}{b^2 - c^2}\frac{2b^2}{c^2} = \frac{2p_c}{\left(1 - c^2/b^2\right)} = \frac{2}{\left(1 - c^2/b^2\right)}\left(p_a - \frac{2\sigma_Y}{\sqrt{3}} \ln c/a\right)$$

$$\boxed{\frac{p_a}{\sigma_Y} = \frac{2\ln c/a}{\sqrt{3}} + \frac{\left(1 - c^2/b^2\right)}{\sqrt{3}}}, \qquad \boxed{p_c = p_a - \frac{2\sigma_Y}{\sqrt{3}} \ln c/a = \frac{\sigma_Y\left(1 - c^2/b^2\right)}{\sqrt{3}}}$$

Hollow Cylinder under Monotonic Pressure

- The constant of integration can be found by noting that the radial displacements in the elastic and plastic regimens must be equal at $r = c$.
- In the plastic region: $u_r = \dfrac{(1+\nu)(1-2\nu)r}{E}\left(\dfrac{2\sigma_Y}{\sqrt{3}}\ln r/a - p_a\right) + \dfrac{C}{r}$
- In the elastic region: $u_r = \dfrac{(1+\nu)p_c c^2 r}{E\left(b^2 - c^2\right)}\left\{(1-2\nu) + \dfrac{b^2}{r^2}\right\}$
- Enforcing the displacement continuity condition

$$C = \frac{2\left(1-\nu^2\right)b^2c^2}{E\left(b^2-c^2\right)}p_c = \frac{2\left(1-\nu^2\right)b^2c^2}{E\left(b^2-c^2\right)}\left(p_a - \frac{2\sigma_Y}{\sqrt{3}}\ln c/a\right)$$

$$= \frac{2\left(1-\nu^2\right)b^2c^2}{E\left(b^2-c^2\right)}\frac{\sigma_Y\left(1-c^2/b^2\right)}{\sqrt{3}} = \frac{2\sigma_Y\left(1-\nu^2\right)c^2}{\sqrt{3}E}$$

Hollow Cylinder under Monotonic Pressure

- **In the plastic region**

  a < r < c

$$\sigma_r = \frac{2\sigma_Y}{\sqrt{3}} \ln r/a - p_a,$$

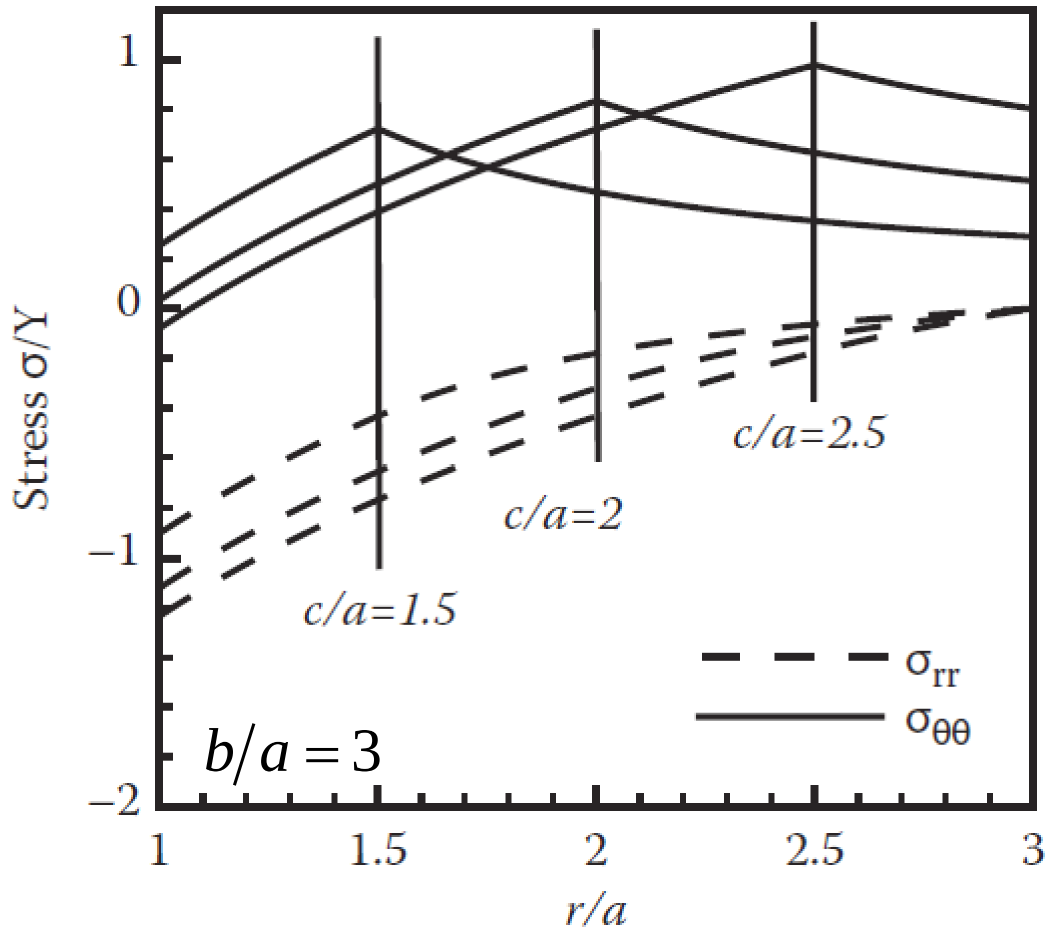
$$\sigma_\theta = \frac{2\sigma_Y}{\sqrt{3}} \ln r/a - p_a + \frac{2\sigma_Y}{\sqrt{3}}$$

- **In the elastic region**

  c < r < b

$$\sigma_r = \frac{\sigma_Y c^2}{\sqrt{3} b^2}\left\{1 - \frac{b^2}{r^2}\right\},$$

$$\sigma_\theta = \frac{\sigma_Y c^2}{\sqrt{3} b^2}\left(1 + \frac{b^2}{r^2}\right)$$



$$\frac{p_a}{\sigma_Y} = \frac{2\ln c/a}{\sqrt{3}} + \frac{\left(1 - c^2/b^2\right)}{\sqrt{3}}, \quad p_c = p_a - \frac{2\sigma_Y}{\sqrt{3}} \ln c/a = \frac{\sigma_Y\left(1 - c^2/b^2\right)}{\sqrt{3}}$$

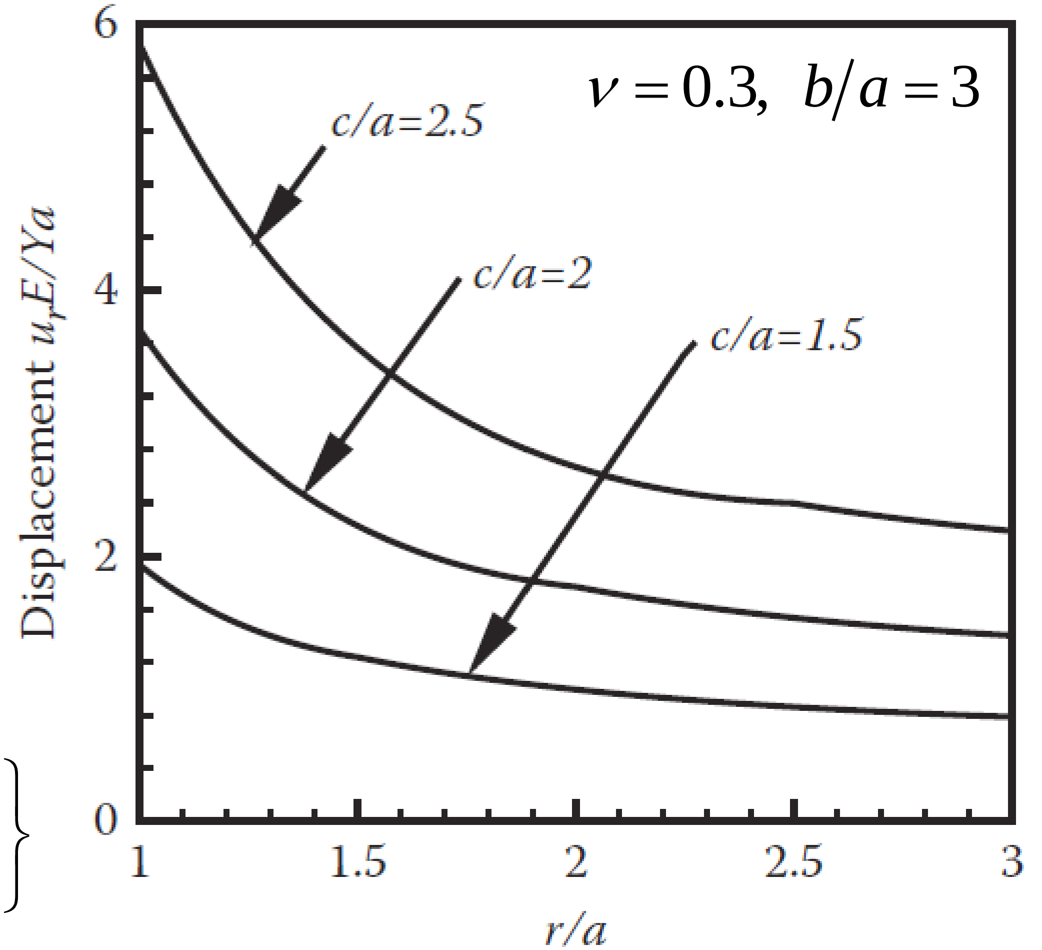
Hollow Cylinder under Monotonic Pressure

- **In the plastic region**
  a* < *r* < *c

$$u_r = \frac{(1+\nu)(1-2\nu)r}{E}\left(\frac{2\sigma_Y}{\sqrt{3}}\ln r/a - p_a\right) + \frac{2\sigma_Y\left(1-\nu^2\right)c^2}{\sqrt{3}Er}$$

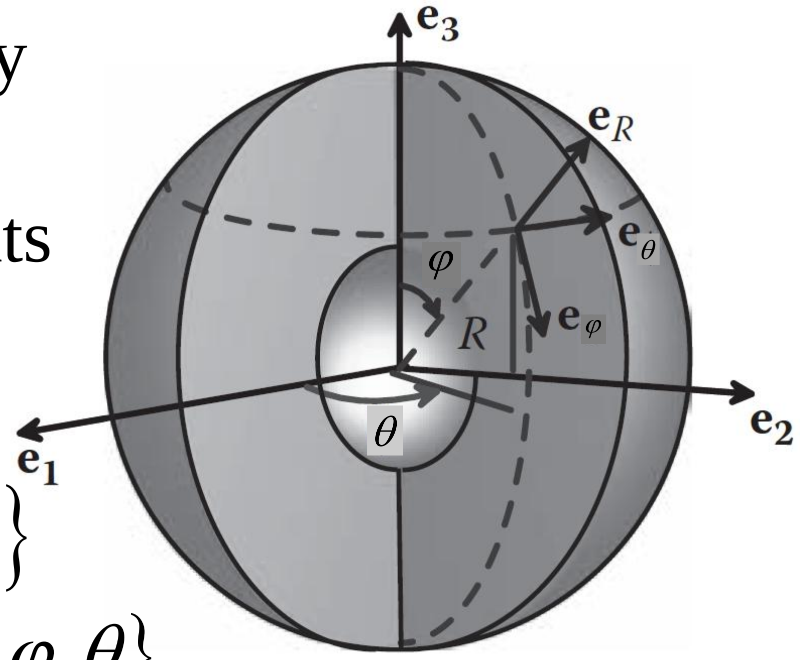
- **In the elastic region**
  c* < *r* < *b

$$u_r = \frac{(1+\nu)\sigma_Y c^2 r}{\sqrt{3}Eb^2}\left\{(1-2\nu)+\frac{b^2}{r^2}\right\}$$



$$\frac{p_a}{\sigma_Y} = \frac{2\ln c/a}{\sqrt{3}} + \frac{\left(1-c^2/b^2\right)}{\sqrt{3}}, \quad p_c = p_a - \frac{2\sigma_Y}{\sqrt{3}}\ln c/a = \frac{\sigma_Y\left(1-c^2/b^2\right)}{\sqrt{3}}$$

Spherically Symmetric Elastoplastic Solids



- Spherically symmetric geometry and loading (i.e., internal body forces, tractions or displacements BCs, nonuniform temperature distribution).
- Spherical-polar bases: $\{\mathbf{e}_R, \mathbf{e}_\varphi, \mathbf{e}_\theta\}$
- Spherical-polar coordinates: $\{R, \varphi, \theta\}$
- Position vector: $\mathbf{x} = R\mathbf{e}_R$
- Displacement vector: $\mathbf{u} = u_R[R]\mathbf{e}_R$
- Body force vector: $\mathbf{F} = F_R[R]\mathbf{e}_R$
- Cauchy stress tensor:

$$\boldsymbol{\sigma} = \sigma_R[R]\mathbf{e}_R\mathbf{e}_R + \sigma_\varphi[R]\mathbf{e}_\varphi\mathbf{e}_\varphi + \sigma_\theta[R]\mathbf{e}_\theta\mathbf{e}_\theta, \quad \sigma_\varphi = \sigma_\theta.$$

Spherically Symmetric Elastoplastic Solids

- Infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = \varepsilon_R[R]\mathbf{e}_R\mathbf{e}_R + \varepsilon_\varphi[R]\mathbf{e}_\varphi\mathbf{e}_\varphi + \varepsilon_\theta[R]\mathbf{e}_\theta\mathbf{e}_\theta, \quad \varepsilon_\varphi = \varepsilon_\theta.$$

- Strain-displacement relation: $\varepsilon_R = \dfrac{\mathrm{d}u_R}{\mathrm{d}R}, \quad \varepsilon_\varphi = \varepsilon_\theta = \dfrac{u_R}{R}$
- Stress-strain relation in elastic region:

$$\begin{Bmatrix} \sigma_R \\ \sigma_\varphi \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_R \\ \varepsilon_\varphi \end{Bmatrix}.$$

- von Mises yield criterion:

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_{ij} - \frac{1}{3}(\sigma_R + 2\sigma_\varphi)\delta_{ij} \Rightarrow \begin{cases} \sigma'_R = \sigma_R - \dfrac{1}{3}(\sigma_R + 2\sigma_\varphi) = \dfrac{2}{3}(\sigma_R - \sigma_\varphi) \\ \sigma'_\varphi = \sigma_\varphi - \dfrac{1}{3}(\sigma_R + 2\sigma_\varphi) = -\dfrac{1}{3}(\sigma_R - \sigma_\varphi) \end{cases}$$

$$\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = |\sigma_R - \sigma_\varphi| = \sigma_Y.$$

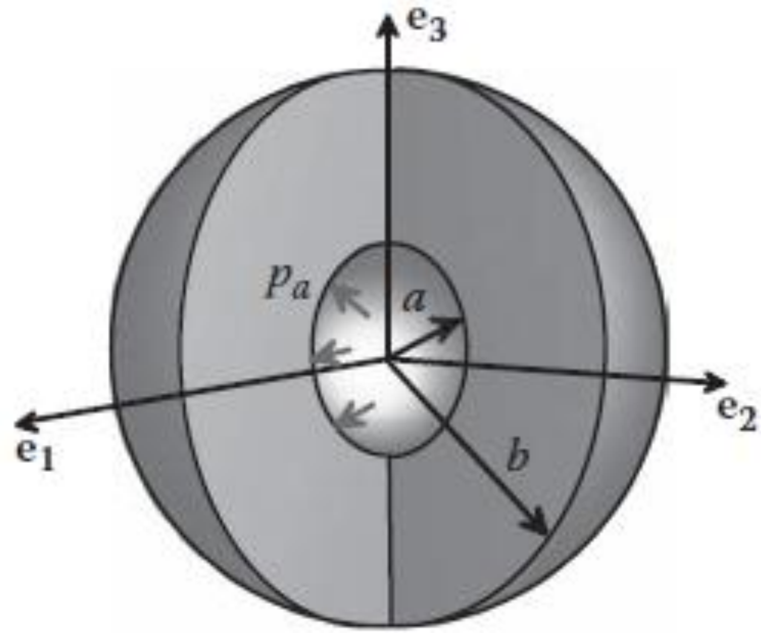
Spherically Symmetric Elastoplastic Solids

- **Stress-strain relation in plastic region**
- Strain partition: $d\varepsilon_R = d\varepsilon_R^e + d\varepsilon_R^p, \;\; d\varepsilon_\varphi = d\varepsilon_\varphi^e + d\varepsilon_\varphi^p, \;\; d\varepsilon_\theta = d\varepsilon_\theta^e + d\varepsilon_\theta^p$
- Elastic strain: $d\varepsilon_R^e = \dfrac{1}{E} d\sigma_R - \dfrac{2\nu}{E} d\sigma_\varphi, \;\; d\varepsilon_\theta^e = d\varepsilon_\varphi^e = \dfrac{1-\nu}{E} d\sigma_\varphi - \dfrac{\nu}{E} d\sigma_R$
- Flow rule:

$$d\varepsilon_{ij}^p = d\bar{\varepsilon}^p \frac{3}{2} \frac{\sigma'_{ij}}{\sigma_Y} \quad \Rightarrow d\varepsilon_R^p = \frac{d\bar{\varepsilon}^p \left(\sigma_R - \sigma_\varphi\right)}{\sigma_Y}, \;\; d\varepsilon_\theta^p = d\varepsilon_\varphi^p = -\frac{d\bar{\varepsilon}^p \left(\sigma_R - \sigma_\varphi\right)}{2\sigma_Y}$$

- Equilibrium equations: $\dfrac{d\sigma_R}{dR} + \dfrac{2}{R}\left(\sigma_R - \sigma_\varphi\right) + F_R = 0.$
- Traction BCs: $\sigma_R\left[a\right] = \sigma_a, \sigma_R\left[b\right] = \sigma_b.$
- Displacement BCs: $u_R\left[a\right] = u_a, u_R\left[b\right] = u_b.$
- There is no clean, direct, and general method for integrating these equations. Instead, solutions must be found using a combination of physical intuition and some algebraic tricks.

Hollow Sphere under Monotonically Increasing Pressure



- We consider a pressurized spherical thick-walled sphere.
- The sphere is stress free before it is loaded.
- No body forces act on the sphere.
- The sphere has uniform temperature.
- The inner surface $R = a$ is subjected to monotonically pressure p_a.
- The outer surface $R = b$ is traction free.
- Strains are infinitesimal.
- We aim to find….

Hollow Sphere under Monotonically Increasing Pressure

- Elastic solution

$$\sigma_R = -\frac{a^3 p_a}{b^3 - a^3}\left(\frac{b^3}{R^3} - 1\right), \quad \sigma_\varphi = \sigma_\theta = \frac{a^3 p_a}{b^3 - a^3}\left(\frac{b^3}{2R^3} + 1\right)$$

- von Mises yield criterion:

$$\sigma_Y = \left|\sigma_R - \sigma_\varphi\right| = \sigma_\varphi - \sigma_R = \frac{a^3 p_a}{b^3 - a^3}\frac{3b^3}{2R^3}$$

- We see that a pressurized elastic sphere first reaches yield at $R = a$, with the elastic limit: $p_a = 2\sigma_Y\left(1 - a^3/b^3\right)/3.$
- If the pressure is increased beyond yield, we anticipate that a region $a < R < c$ will deform plastically, whereas a region $c < R < b$ remains elastic.

Hollow Sphere under Monotonically Increasing Pressure

- **In the plastic region *a* < *R* < *c***
- von Mises yield criterion

$$\sigma_Y = \left|\sigma_R - \sigma_\varphi\right| = \sigma_\varphi - \sigma_R = \frac{a^3 p_a}{b^3 - a^3}\frac{3b^3}{2R^3}$$

- Equation of static equilibrium

$$\frac{d\sigma_R}{dR} + \frac{2}{R}\left(\sigma_R - \sigma_\varphi\right) + F_R = 0, \quad \sigma_\varphi - \sigma_R = \sigma_Y \quad \Rightarrow \frac{d\sigma_R}{dR} - \frac{2\sigma_Y}{R} = 0$$

- Integrate and apply the BCs at $R = a$

$$\sigma_R = 2\sigma_Y \ln R/a - p_a \quad \Rightarrow \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y$$

Hollow Sphere under Monotonically Increasing Pressure

- **In the plastic region *a* < *R* < *c***
- Elastic strain: $d\varepsilon_R^e = \frac{1}{E} d\sigma_R - \frac{2\nu}{E} d\sigma_\varphi, \quad d\varepsilon_\theta^e = d\varepsilon_\varphi^e = \frac{1-\nu}{E} d\sigma_\varphi - \frac{\nu}{E} d\sigma_R$
- The plastic strains satisfy: $d\varepsilon_R^p + 2d\varepsilon_\varphi^p = 0$
- The elastic strains thus satisfy

$$d\varepsilon_R + 2d\varepsilon_\varphi = d\varepsilon_R^e + 2d\varepsilon_\varphi^e = \frac{1-2\nu}{E}\left(d\sigma_R + 2d\sigma_\varphi\right)$$

- Since the pressure is monotonically increasing, the incremental stress-strain relation can be integrated

$$\Rightarrow \varepsilon_R^e + 2\varepsilon_\varphi^e = \frac{1-2\nu}{E}\left(\sigma_R + 2\sigma_\varphi\right) \quad \Rightarrow \frac{du_R}{dR} + \frac{2u_R}{R} = \frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right)$$

$$= \frac{1-2\nu}{E}\left(6\sigma_Y \ln R/a - 3p_a + 2\sigma_Y\right) \quad \Rightarrow u_R = -\frac{1-2\nu}{E} R\left(p_a - 2\sigma_Y \ln R/a\right) + \frac{C}{R^2}$$

- The constant of integration, C, will be determined later.

Hollow Sphere under Monotonically Increasing Pressure

- **In the elastic region *c* < *R* < *b***

$$\sigma_R = -\frac{c^3 p_c}{b^3 - c^3}\left(\frac{b^3}{R^3} - 1\right), \quad \sigma_\varphi = \sigma_\theta = \frac{c^3 p_c}{b^3 - c^3}\left(\frac{b^3}{2R^3} + 1\right), \quad u_R = \frac{(1+\nu)}{E\left(b^3 - c^3\right)} R \left(\frac{b^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right) c^3 p_c$$

- Form the radial stress in the plastic region, we obtain the pressure at the elastic-plastic boundary $R = c$

$$\sigma_R[c] = 2\sigma_Y \ln c/a - p_a = -p_c \qquad \Rightarrow p_c = p_a - 2\sigma_Y \ln c/a$$

- The elastic-plastic boundary is located by noting that the stress in the elastic region must just reach yield at $R = c$.

$$\sigma_Y = \sigma_\varphi - \sigma_R = \frac{3 p_c b^3}{2\left(b^3 - c^3\right)} = \frac{3\left(p_a - 2\sigma_Y \ln c/a\right) b^3}{2\left(b^3 - c^3\right)}$$

$$\Rightarrow \boxed{\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1 - c^3/b^3\right)}, \qquad \boxed{p_c = p_a - 2\sigma_Y \ln c/a = \frac{2\sigma_Y}{3}\left(1 - c^3/b^3\right)}$$

Hollow Sphere under Monotonically Increasing Pressure

- The constant of integration can be found by noting that the radial displacements in the elastic and plastic regimens must be equal at $R = c$.
- In the plastic region: $u_R = -\dfrac{1-2\nu}{E} R\left(p_a - 2\sigma_Y \ln R/a\right) + \dfrac{C}{R^2}$
- In the elastic region: $u_R = \dfrac{(1+\nu)}{E\left(b^3 - c^3\right)} R\left(\dfrac{b^3}{2R^3} + \dfrac{1-2\nu}{1+\nu}\right) c^3 p_c$
- Enforcing the displacement continuity condition

$$C = \frac{3(1-\nu)c^3 b^3}{2E\left(b^3 - c^3\right)} p_c = \frac{3(1-\nu)c^3 b^3}{2E\left(b^3 - c^3\right)}\left(p_a - 2\sigma_Y \ln c/a\right)$$

$$= \frac{3(1-\nu)c^3 b^3}{2E\left(b^3 - c^3\right)} \frac{2\sigma_Y}{3}\left(1 - c^3/b^3\right) = \frac{\sigma_Y (1-\nu)c^3}{E}.$$

Hollow Sphere under Monotonically Increasing Pressure

- **In the plastic region** $a < R < c$

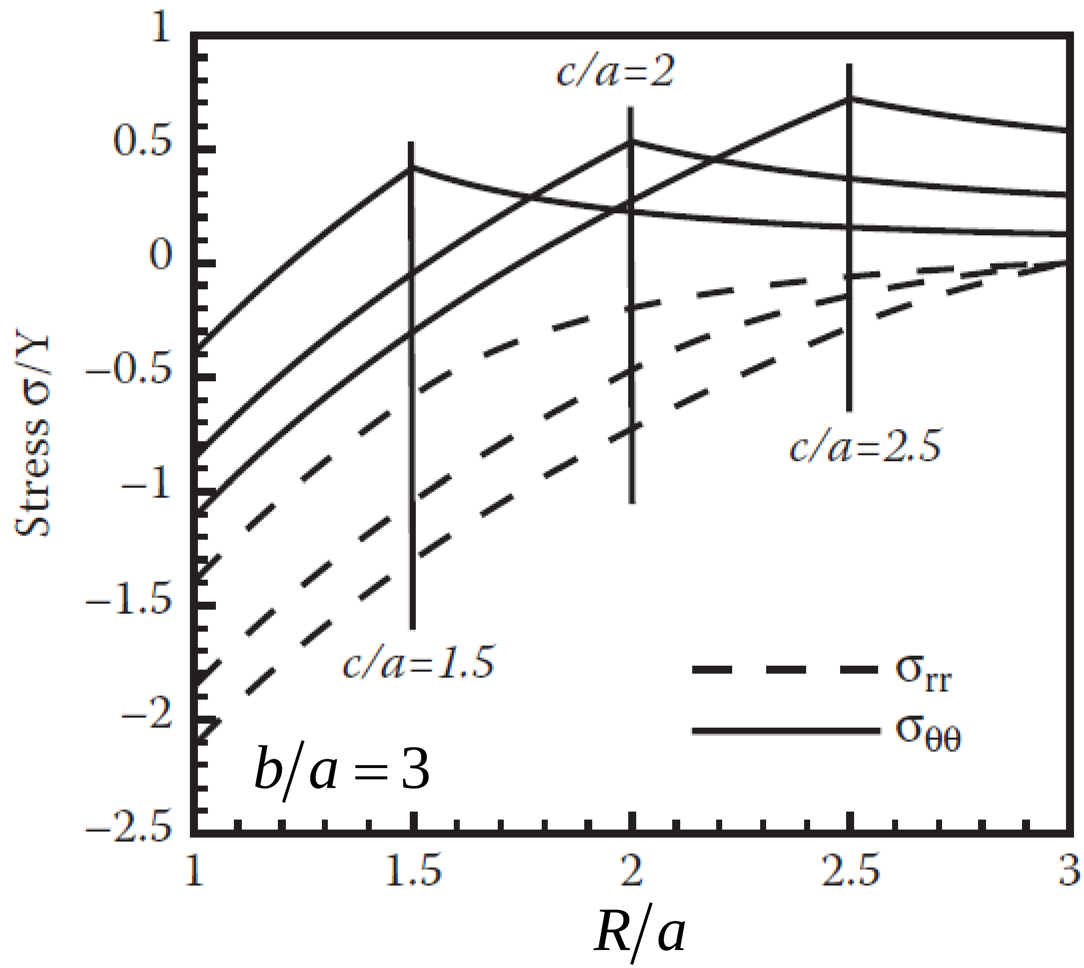
$$\sigma_R = 2\sigma_Y \ln R/a - p_a,$$

$$\sigma_\theta = 2\sigma_Y \ln R/a - p_a + \sigma_Y$$

- **In the elastic region** $c < R < b$

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3} - 1\right),$$

$$\sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right)$$



$$\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1 - c^3/b^3\right), \quad p_c = p_a - 2\sigma_Y \ln c/a = \frac{2\sigma_Y}{3}\left(1 - c^3/b^3\right)$$

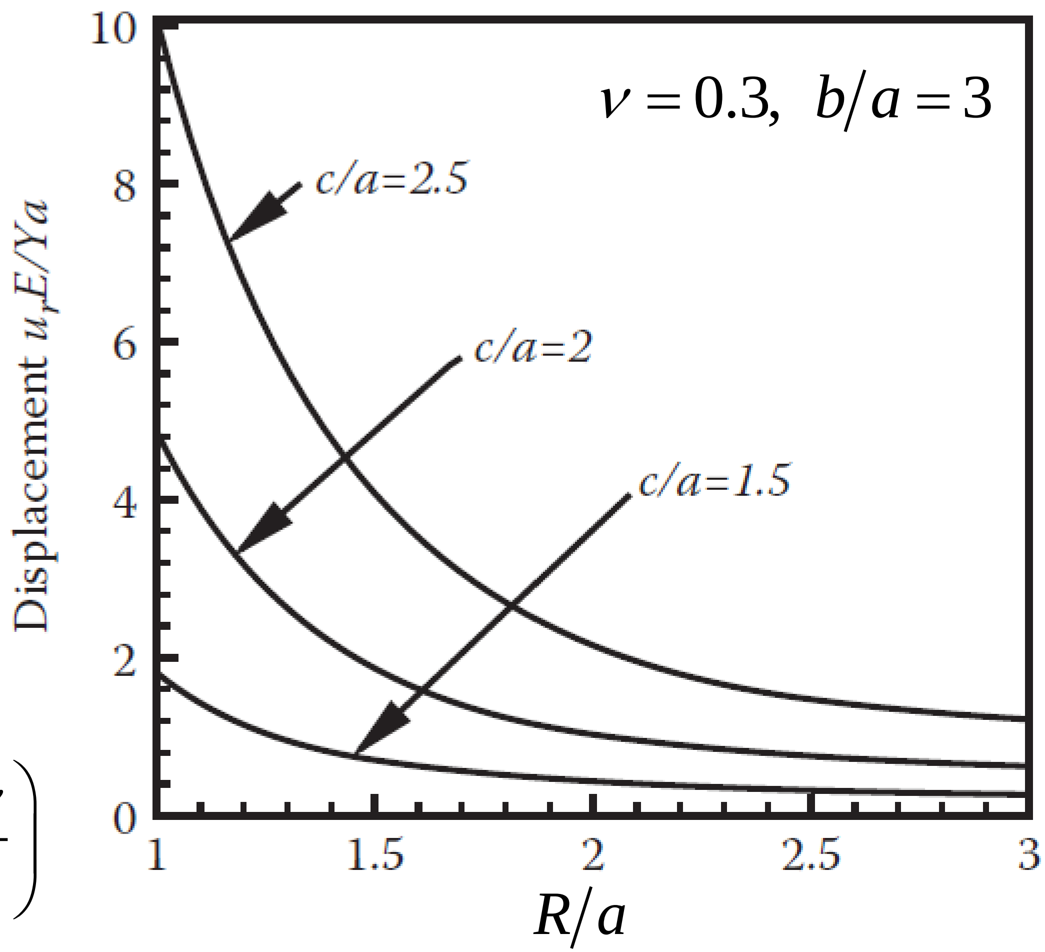
Hollow Sphere under Monotonically Increasing Pressure

- **In the plastic region**
  a < R < c

$$u_R = -\frac{1-2\nu}{E} R\left(p_a - 2\sigma_Y \ln R/a\right) + \frac{\sigma_Y \left(1-\nu\right) c^3}{ER^2}$$

- **In the elastic region**
  c < R < b

$$u_R = \frac{2\sigma_Y \left(1+\nu\right) c^3}{3Eb^3} R\left(\frac{b^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right)$$



$$\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1 - c^3/b^3\right), \quad p_c = p_a - 2\sigma_Y \ln c/a = \frac{2\sigma_Y}{3}\left(1 - c^3/b^3\right)$$

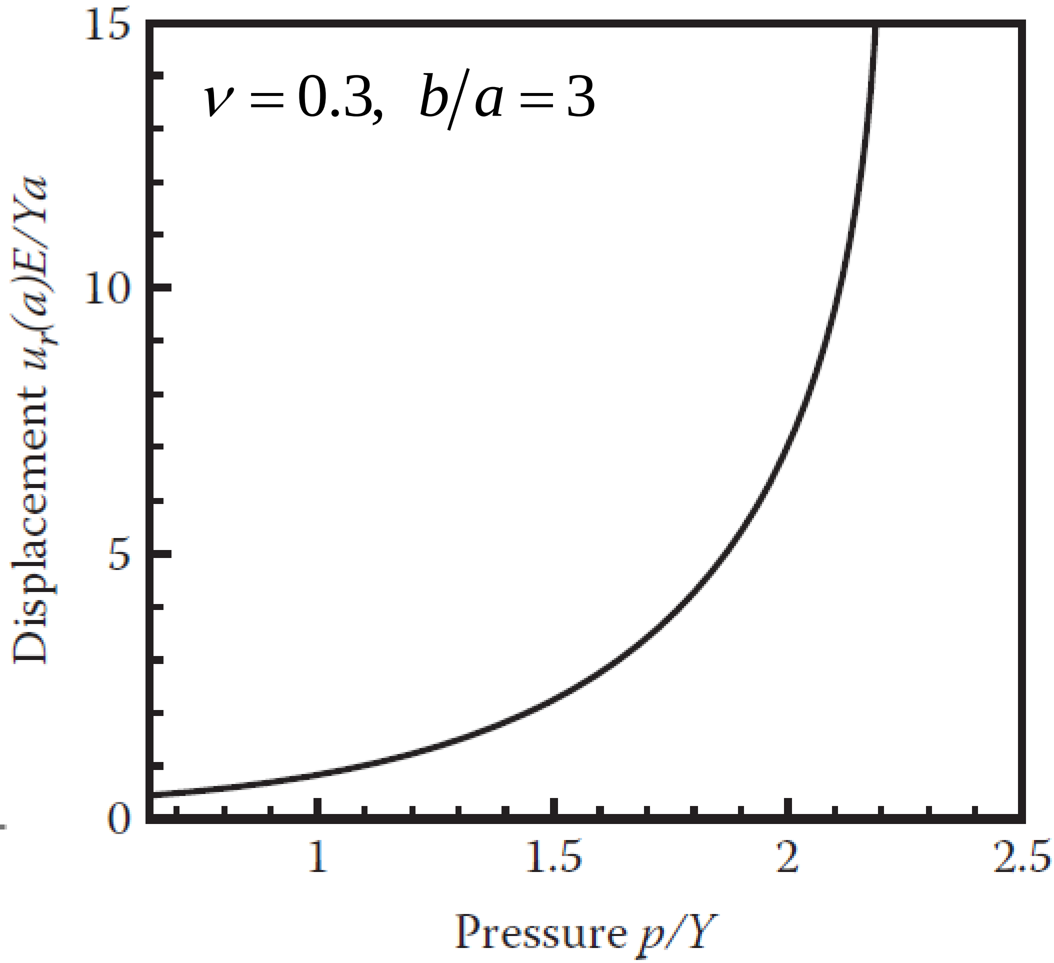
Hollow Sphere under Monotonically Increasing Pressure

- **In the plastic region $a < R < c$**

$$u_R = -\frac{1-2\nu}{E} R\left(p_a - 2\sigma_Y \ln R/a\right) + \frac{\sigma_Y\left(1-\nu\right)c^3}{ER^2}$$

- **At the interior surface $R = a$**

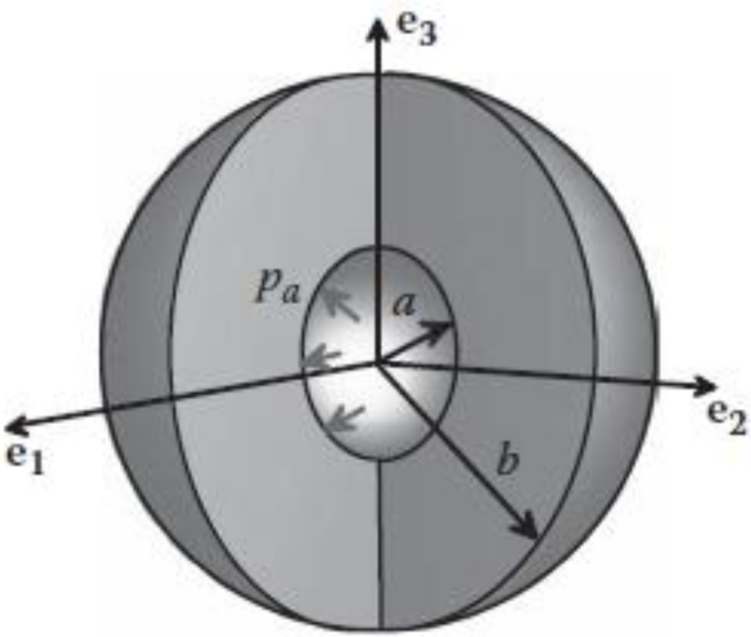
$$u_R\left[a\right] = -\frac{1-2\nu}{E} a p_a + \frac{\sigma_Y\left(1-\nu\right)c^3}{Ea^2}$$



$$\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1 - c^3/b^3\right), \quad p_c = p_a - 2\sigma_Y \ln c/a = \frac{2\sigma_Y}{3}\left(1 - c^3/b^3\right)$$

Hollow Sphere under Cyclic Internal Pressure

- The sphere is stress free before it is loaded.
- No body forces act on the sphere.
- The sphere has uniform temperature distribution
- The outer surface $R = b$ is traction free.



- The inner surface of the sphere $R = a$ is repeatedly subjected to pressure p_a and then unloaded to zero pressure.
- **The nature of the solution depends on the magnitude of the internal pressure.**

Hollow Sphere under Cyclic Internal Pressure

- **If** $p_a \le 2\sigma_Y\left(1-a^3/b^3\right)/3$
- **the maximum value of p_a applied to the sphere does not exceed the elastic limit, the solid remains elastic throughout the loading cycle.**
- The sphere is stress free after unloading and remains elastic throughout all subsequent load cycles.

- **From the previous case study for monotonically increasing load, in the plastic region $a < R < c$**

$$\sigma_R = 2\sigma_Y \ln R/a - p_a, \quad \sigma_\theta = 2\sigma_Y \ln R/a - p_a + \sigma_Y$$

- **From the limiting case of a completely yielded shell ($c = b$), the collapse load can be determined**

$$\sigma_R[b] = 0 = 2\sigma_Y \ln b/a - p_a, \quad \Rightarrow p_a = 2\sigma_Y \ln b/a$$

Hollow Sphere under Cyclic Internal Pressure

- **Practical pressure range:** $2\sigma_Y\left(1-a^3/b^3\right)/3 \le p_a \le 2\sigma_Y \ln b/a$

- For pressures in this range, the region between $R = a$ and $R = c$ deforms plastically during the first application of pressure, whereas the region between $c < R < b$ remains elastic.

$$\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1-c^3/b^3\right), \quad p_c = p_a - 2\sigma_Y \ln c/a = \frac{2\sigma_Y}{3}\left(1-c^3/b^3\right)$$

- In this case, the solid is permanently deformed. After unloading, its internal and external radii are slightly increased, and the sphere is in a state of residual stress.
- The applied pressure p_a cannot exceed the collapse load.

Hollow Sphere under Cyclic Internal Pressure

- **Practical pressure range:** $2\sigma_Y\left(1-a^3/b^3\right)/3 \le p_a \le 2\sigma_Y \ln b/a$
- At the maximum pressure of the first cycle
- In the plastic region $a < R < c$

$$\sigma_R = 2\sigma_Y \ln R/a - p_a, \quad \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y$$

- In the elastic region $c < R < b$

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3}-1\right), \quad \sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3}+1\right)$$

- **If the residual stress is less than or just reaches the yield stress at $R = a$ when the pressure is reduced to zero after the first unloading, i.e. for $a < R < c$**

$$\sigma_R = 2\sigma_Y \ln R/a - p_a - \frac{a^3 p_a}{b^3-a^3}\left(1-\frac{b^3}{R^3}\right), \quad \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y - \frac{a^3 p_a}{b^3-a^3}\left(\frac{b^3}{2R^3}+1\right)$$

Hollow Sphere under Cyclic Internal Pressure

- After unloading of the first cycle (at zero pressure)

$$\Rightarrow \sigma_R - \sigma_\varphi = \frac{a^3 p_a}{b^3 - a^3}\frac{3b^3}{2R^3} - \sigma_Y \qquad \Rightarrow \sigma_R[a] - \sigma_\varphi[a] = \frac{3}{2}\frac{p_a}{\left(1 - a^3/b^3\right)} - \sigma_Y$$

$$p_a \ge 2\sigma_Y\left(1 - a^3/b^3\right)/3 \quad \Rightarrow \sigma_R[a] - \sigma_\varphi[a] > 0$$

$$\sigma_R[a] - \sigma_\varphi[a] = \frac{3}{2}\frac{p_a}{\left(1 - a^3/b^3\right)} - \sigma_Y \le \sigma_Y \qquad \Rightarrow \boxed{p_a \le 4\sigma_Y\left(1 - a^3/b^3\right)/3}$$

- The maximum load is known as the **shakedown** limit.
- In this case (after unloading of the first cycle), stresses in the **elastic** region $c < R < b$ are

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3} - 1\right) - \frac{a^3 p_a}{b^3 - a^3}\left(1 - \frac{b^3}{R^3}\right), \quad \sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right) - \frac{a^3 p_a}{b^3 - a^3}\left(\frac{b^3}{2R^3} + 1\right)$$

Hollow Sphere under Cyclic Internal Pressure

- If $2\sigma_Y\left(1-a^3/b^3\right)/3 \le p_a \le 4\sigma_Y\left(1-a^3/b^3\right)/3$
- the cylinder deforms plastically during the first application of pressure. It then deforms elastically (**no yield**) while the pressure is removed.
- During subsequent pressure cycles between zero and the maximum pressure, the cylinder deforms elastically.
- Residual stresses introduced during the first loading cycle are protective and prevent additional plasticity. This behavior is known as **shakedown**.

Hollow Sphere under Cyclic Internal Pressure

- If $4\sigma_Y\left(1-a^3/b^3\right)/3 \le p_a \le 2\sigma_Y \ln b/a$
- **At the maximum pressure of the first cycle**
- In the plastic region $a < R < c$

$$\sigma_R = 2\sigma_Y \ln R/a - p_a, \quad \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y$$

- In the elastic region $c < R < b$

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3} - 1\right), \quad \sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right)$$

Hollow Sphere under Cyclic Internal Pressure

- **Consider residual stress that is larger than the yield stress for $a < R < d$ ($< c$) when the pressure is reduced to zero after the first unloading.**
- Therefore, this is a plastic zone as the pressure is reduced to zero. During subsequent cycles of loading, this region is repeatedly plastically deformed, stretching in the hoop direction during increasing pressure and compressing as the pressure is reduced to zero.
- Anticipate the yield condition

$$\sigma_R > 0, \sigma_\varphi < 0 \quad \Rightarrow \sigma_R - \sigma_\varphi = \sigma_Y .$$

Hollow Sphere under Cyclic Internal Pressure

- Equilibrium condition

$$\frac{d\sigma_R}{dR}+\frac{2}{R}\left(\sigma_R-\sigma_\varphi\right)=0 \quad \Rightarrow \frac{d\sigma_R}{dR}=-\frac{2\sigma_Y}{R}$$

$$\sigma_R\left[a\right]=0 \quad \Rightarrow \boxed{\sigma_R=-2\sigma_Y \ln R/a} \quad \Rightarrow \boxed{\sigma_\varphi=\sigma_R-\sigma_Y=-2\sigma_Y \ln R/a-\sigma_Y}$$

- **In the shakedown region *d* < *R* < *c***
- This region deforms plastically during the first cycle of pressure but remains elastic for all subsequent cycles.
- This is a "shakedown region."
- The change in stress during unloading can be calculated by regarding the region $d < R < b$ as a spherical shell, subjected to radial pressure at $R = d$.
- At the maximum load: $p'_d=-\sigma_R\left[d\right]=p_a-2\sigma_Y \ln d/a$.

Hollow Sphere under Cyclic Internal Pressure

- After unloading: $p_d = -\sigma_R[d] = 2\sigma_Y \ln d/a.$
- The change in pressure: $\Delta p_d = p_d - p_d' = 4\sigma_Y \ln d/a - p_a.$
- We can simply add the elastic stress induced by this pressure change to the stress at maximum load.

$$\sigma_R = 2\sigma_Y \ln R/a - p_a + \frac{d^3 \Delta p_d}{b^3 - d^3}\left(1 - \frac{b^3}{R^3}\right), \quad \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y + \frac{d^3 \Delta p_d}{b^3 - d^3}\left(\frac{b^3}{2R^3} + 1\right)$$

- **The boundary of the cyclic plastic zone** is determined by the condition that the stress in the shakedown regime must just reach yield at $R = d$ when the pressure reaches zero.

$$\sigma_Y = \sigma_R[d] - \sigma_\varphi[d] = -\frac{3\Delta p_d}{2\left(1 - d^3/b^3\right)} - \sigma_Y \quad \Rightarrow \boxed{\Delta p_d = -4\sigma_Y\left(1 - d^3/b^3\right)/3} = 4\sigma_Y \ln d/a - p_a$$

$$\Rightarrow \boxed{p_a = 4\sigma_Y\left(1 - d^3/b^3\right)/3 + 4\sigma_Y \ln d/a}$$

Hollow Sphere under Cyclic Internal Pressure

- With Δp_d, the stress in the shakedown region ($d < R < c$) can further be simplified

$$\sigma_R = 2\sigma_Y \ln R/a - p_a - \frac{4\sigma_Y d^3}{3b^3}\left(1 - \frac{b^3}{R^3}\right), \quad \sigma_\varphi = 2\sigma_Y \ln R/a - p_a + \sigma_Y - \frac{4\sigma_Y d^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right)$$

- **In the elastic region *c* < *R* < *b***
- This region experiences elastic cycles of strain. The solution in this region is derived in the same way as the solution for the shakedown region, except that the stress at the maximum load is given by solutions for $c < R < b$.

Hollow Sphere under Cyclic Internal Pressure

- The change in stress during unloading can be calculated by regarding the region $c < R < b$ as a spherical shell, subjected to radial pressure at $R = c$.
- At the maximum load: $p'_c = -\sigma_R[c] = \dfrac{2\sigma_Y c^3}{3b^3}\left(\dfrac{b^3}{c^3} - 1\right)$
- After unloading: $p_c = -\sigma_R[c] = -2\sigma_Y \ln c/a + p_a + \dfrac{4\sigma_Y d^3}{3b^3}\left(1 - \dfrac{b^3}{c^3}\right)$
- The change in pressure at $R = c$

$$\Delta p_c = p_c - p'_c = p_a - 2\sigma_Y \ln c/a + \frac{4\sigma_Y d^3}{3b^3}\left(1 - \frac{b^3}{c^3}\right) + \frac{2\sigma_Y c^3}{3b^3}\left(1 - \frac{b^3}{c^3}\right) = \frac{4\sigma_Y d^3}{3b^3}\left(1 - \frac{b^3}{c^3}\right)$$

$$\frac{p_a}{\sigma_Y} = 2\ln c/a + \frac{2}{3}\left(1 - c^3/b^3\right) \quad \Rightarrow p_a - 2\sigma_Y \ln c/a + \frac{2\sigma_Y c^3}{3b^3}\left(1 - \frac{b^3}{c^3}\right) = 0$$

Hollow Sphere under Cyclic Internal Pressure

- We then add the elastic stress induced by this pressure change to the stress at the maximum load ($c < R < b$).

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3} - 1\right) + \frac{c^3 \Delta p_c}{b^3 - c^3}\left(1 - \frac{b^3}{R^3}\right), \quad \sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right) + \frac{c^3 \Delta p_c}{b^3 - c^3}\left(\frac{b^3}{2R^3} + 1\right)$$

- With Δp_c, the stress in the elastic region ($c < R < b$) can further be simplified

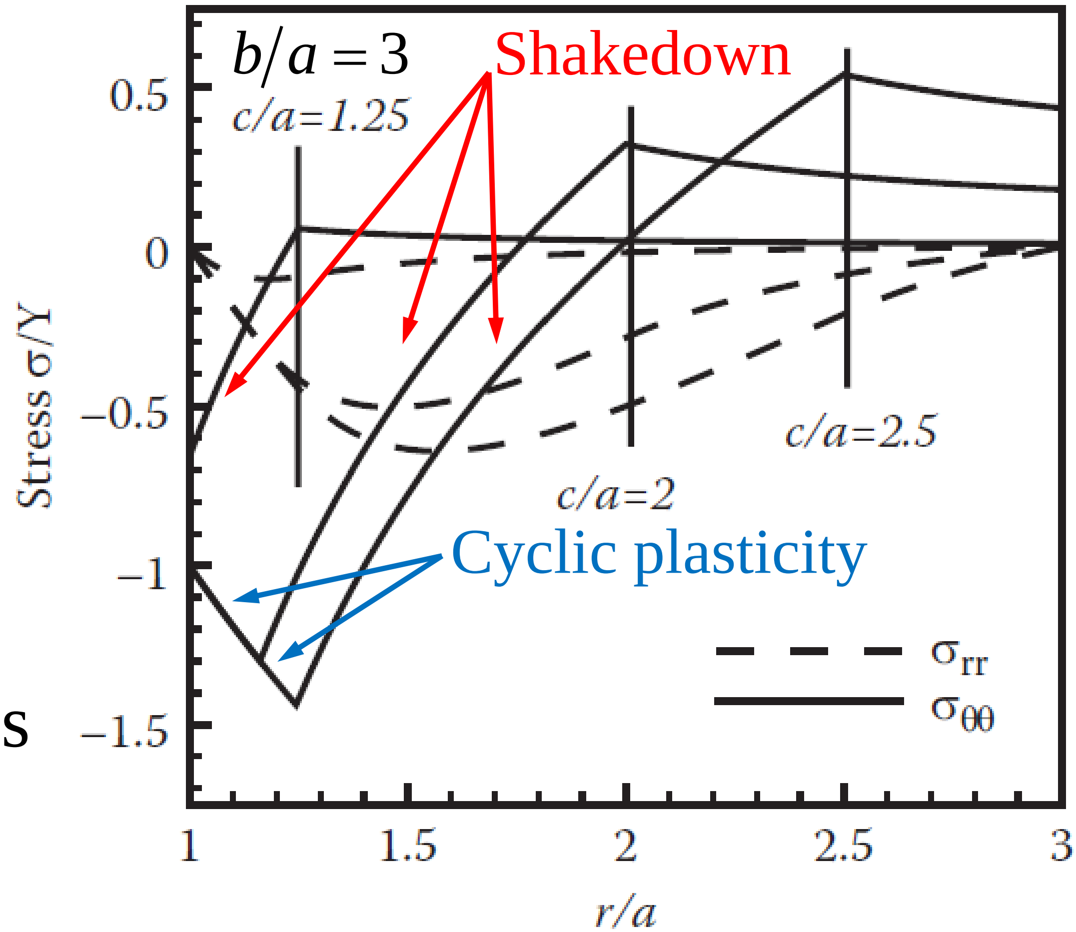
$$\Delta p_c = \frac{4\sigma_Y d^3}{3b^3}\left(1 - \frac{b^3}{c^3}\right) \Rightarrow \frac{c^3 \Delta p_c}{b^3 - c^3} = -\frac{4\sigma_Y d^3}{3b^3}$$

$$\sigma_R = -\frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{R^3} - 1\right) - \frac{4\sigma_Y d^3}{3b^3}\left(1 - \frac{b^3}{R^3}\right), \quad \sigma_\varphi = \frac{2\sigma_Y c^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right) - \frac{4\sigma_Y d^3}{3b^3}\left(\frac{b^3}{2R^3} + 1\right)$$

- In the preceding discussion, we have assumed that the cylinder is thick enough to support an arbitrarily large pressure, without exceeding the collapse load
- For thinner-walled spheres, some regimens will be inaccessible.

Hollow Sphere under Cyclic Internal Pressure

- The solution for *c*/*a* = 1.25 is below the shakedown limit.
- The residual stresses are predominantly compressive.
- Bolt holes, pressure vessels, and gun barrels are often purposely pressurized above the elastic limit so as to prevent crack propagation.



Residual stress distributions after unloading, for various positions of the elastic/plastic boundary *c*/*a*.