Strain Measures

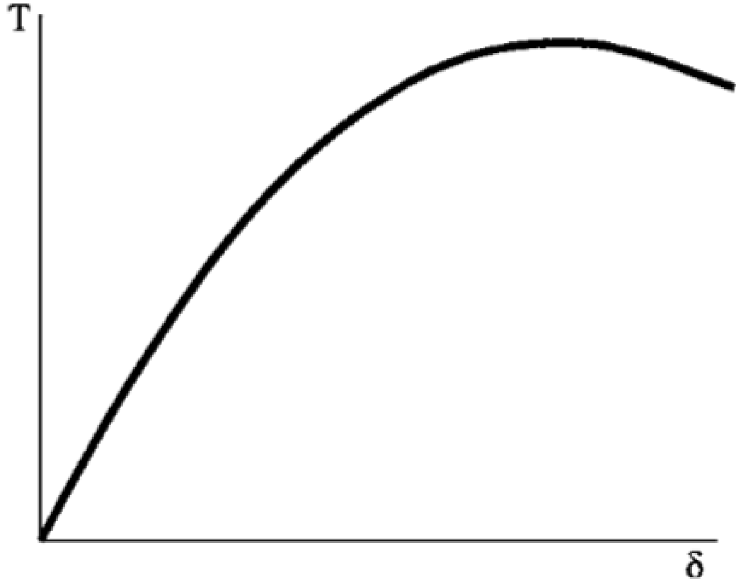
mi@seu.edu.cn

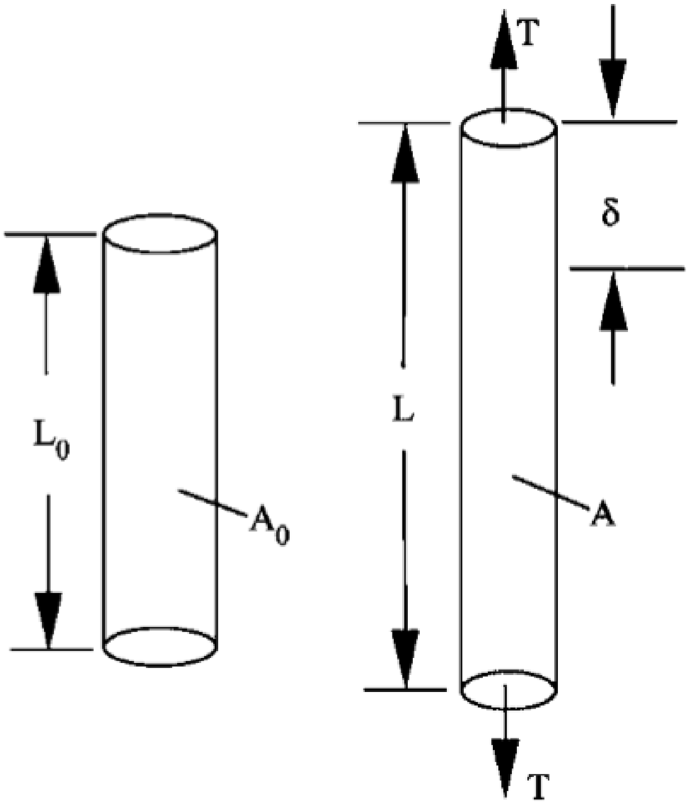
Outline

- Concept of strain（应变的概念）
- Deformation and displacement gradient（变形和位移梯度）
- Cauchy-Green strain tensors（C-G应变）
- Polar decomposition（极分解）
- Jacobian of deformation（变形雅可比）
- Different measures of strain （应变度量）
- Simple deformations（简单变形）
- Small strain theory（小应变理论）
- Material and spatial time derivatives（材料时间导数和空间时间导数）
- Stretch rate and spin rate（变形率和转动率）

Concept of Strain

- Elongation: δ
- Percentage of elongation: $\varepsilon = \delta / l_0$
- Stretch: $\lambda = l/l_0 = 1 + \varepsilon$





Different Measures of Strain

- Engineering strain:
- True strain:
- Difference in length square:
- Difference in length square:
- Logarithmic strain:

$$e_1 = \frac{\delta}{l_0};$$

$$e_2 = \frac{\delta}{l} = \frac{\delta}{l_0+\delta} = \frac{\delta / l_0}{1+\delta / l_0} = \delta / l_0 - \left(\delta / l_0\right)^2 + \ldots;$$

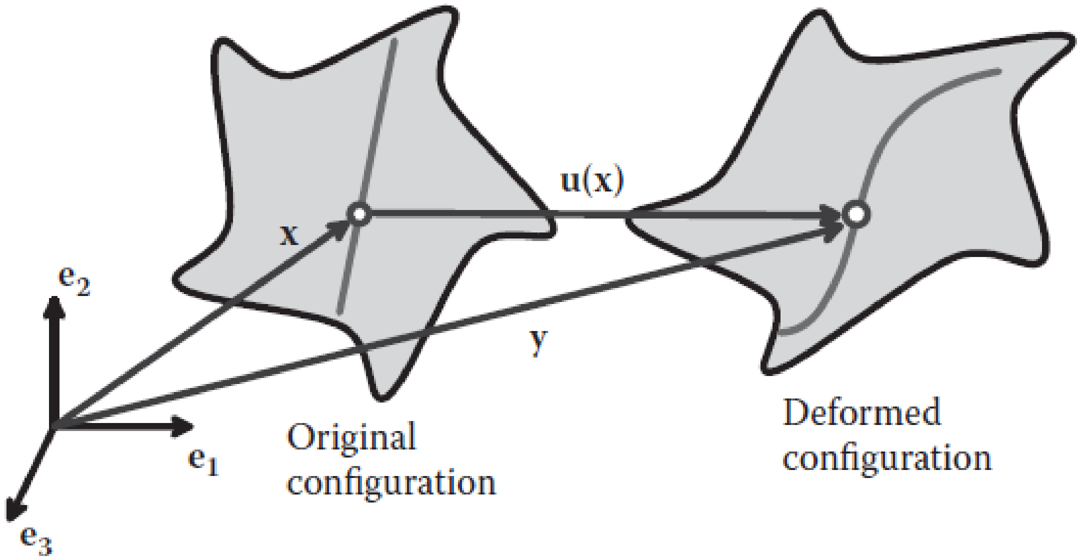
$$e_3 = \frac{l^2 - l_0^2}{2l_0^2} = \frac{\left(l+l_0\right)\left(l-l_0\right)}{2l_0^2} = \frac{\left(2l_0+\delta\right)\left(\delta\right)}{2l_0^2} = \frac{\delta}{l_0} + \frac{\delta^2}{2l_0^2};$$

$$e_4 = \frac{l^2 - l_0^2}{2l^2} = \frac{\left(l+l_0\right)\left(l-l_0\right)}{2l^2} = \frac{\left(2l_0+\delta\right)\left(\delta\right)}{2\left(l_0+\delta\right)^2};$$

$$e_5 = \int_{l_0}^{l} \frac{dl}{l} = \ln\frac{l}{l_0} = \ln\left(1+\delta / l_0\right) = \delta / l_0 - \frac{1}{2}\left(\delta / l_0\right)^2 + \ldots;$$

- All these measures are equivalent for small elongation and thus equivalent from an engineering point of view.
- How to generalize these to 3D?

Deformation Gradient Tensor



$$\vec{u} = \vec{y} - \vec{x} \quad \text{(displacement vector)}$$

$$\vec{y} = \vec{y}(\vec{x}, t)$$

$$y_i = y_i(x_1, x_2, x_3, t)$$

$$\mathrm{d}y_i = \frac{\partial y_i}{\partial x_j}\mathrm{d}x_j = F_{ij}\mathrm{d}x_j$$

$$v_i(x_k, t) = \frac{\partial y_i}{\partial t} = \left.\frac{\partial u_i(x_k, t)}{\partial t}\right|_{x_k=\text{const}}$$

- Deformation gradient:

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

- We wish to find a measure of strain, a relative measure of how material points move with respect to each other, that is independent of rigid body rotation.

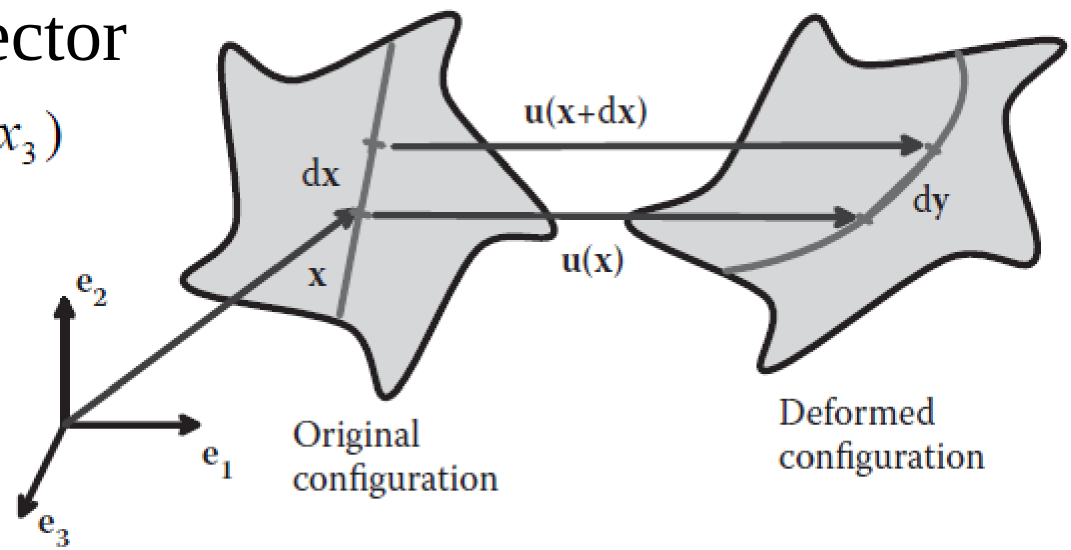
Displacement Vector

- Consider an arbitrary fiber
- Define displacement vector

$$\vec{y} = \vec{y}(x_1, x_2, x_3) = \vec{x} + \vec{u}(x_1, x_2, x_3)$$

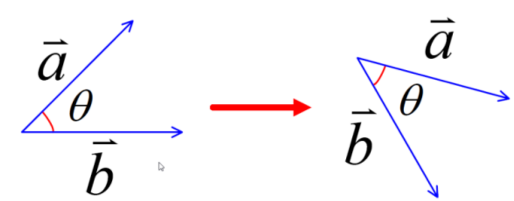
- Deformation gradient contains information about both stretch and rotation:

$$d\vec{x} = \vec{m}\,dl_0 \qquad\qquad d\vec{y} = \vec{n}\,dl$$



$$d\vec{y} = \vec{n}\,dl = F\vec{m}\,dl_0 \qquad \Rightarrow \underline{F}\vec{m} = \vec{n}\,dl/dl_0 \qquad \Rightarrow \underline{F}\vec{m} = \lambda\vec{n}$$

- In order to separate stretch from rigid body rotation, consider the dot product of two fibers.
- Since the dot product only depends on the relative angle between the two vectors, rigid body rotation can be been effectively “filtered” out.



Cauchy-Green Strain Tensors

- Consider the **dot product** of two differential segments in both the undeformed and deformed configurations

$$d\bar{y}_1 \cdot d\bar{y}_2 = \underline{F}\,d\bar{x}_1 \cdot \underline{F}\,d\bar{x}_2 = d\bar{x}_1 \cdot \underline{F}^{\mathrm{T}}\,\underline{F}\,dx_2$$

$$d\bar{x}_1 \cdot d\bar{x}_2 = \underline{F}^{-1} d\bar{y}_1 \cdot \underline{F}^{-1} d\bar{y}_2 = d\bar{y}_1 \cdot \underline{F}^{-\mathrm{T}}\,\underline{F}^{-1} d\bar{y}_2 = d\bar{y}_1 \cdot \left(\underline{F}\,\underline{F}^{\mathrm{T}}\right)^{-1} d\bar{y}_2$$

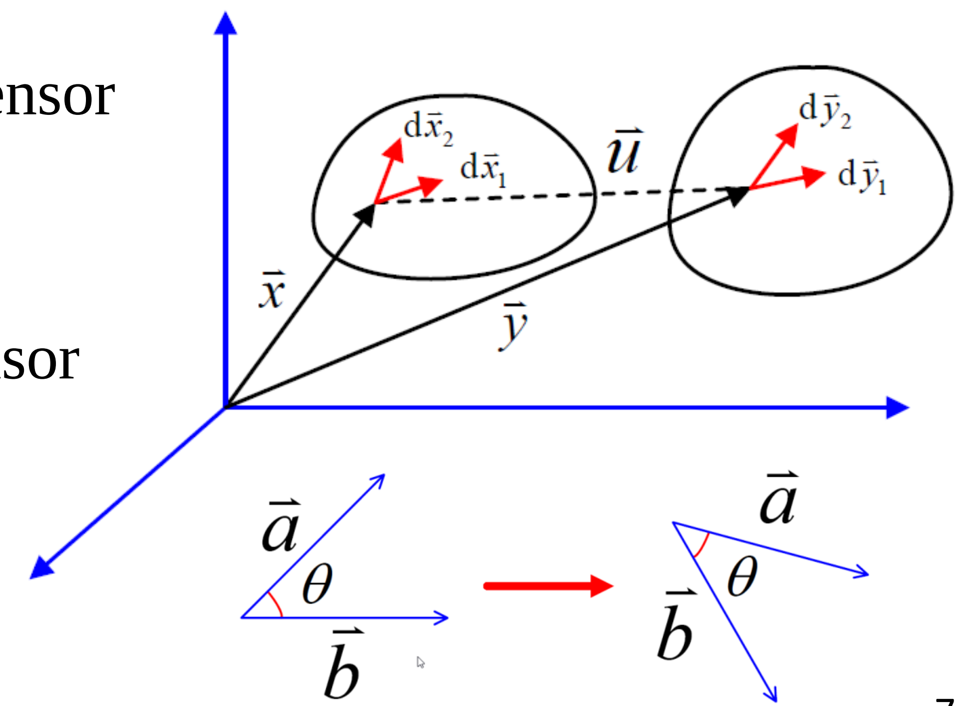
- Right Cauchy-Green strain tensor

$$\underline{C} = \underline{F}^T\,\underline{F}\,, \quad C_{ij} = F_{ki}F_{kj}$$

- Left Cauchy-Green strain tensor

$$\underline{B} = \underline{F}\,\underline{F}^T\,, \quad B_{ij} = F_{ik}F_{jk}$$

- Both are symmetric.



Physical/Geometric Interpretations of C

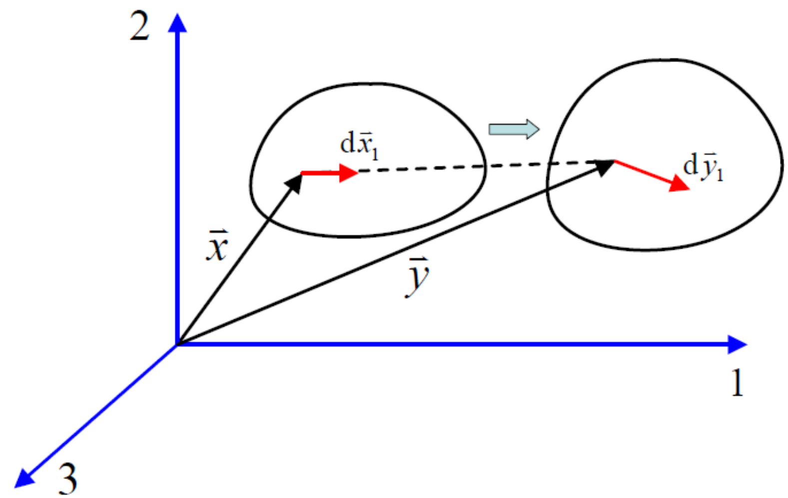
- General matrix form

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

- Principal values/directions

$$\underline{C} = \begin{bmatrix} C_{I} & 0 & 0 \\ 0 & C_{II} & 0 \\ 0 & 0 & C_{III} \end{bmatrix}, \quad \vec{m}_I \perp \vec{m}_{II} \perp \vec{m}_{III}$$

- Consider a fiber initially along one of the base vectors



$$\mathrm{d}\bar{x}_1 = \mathrm{d}l_{10}\bar{e}_1, \quad \mathrm{d}\bar{y}_1 = \mathrm{d}l_1\vec{n}$$

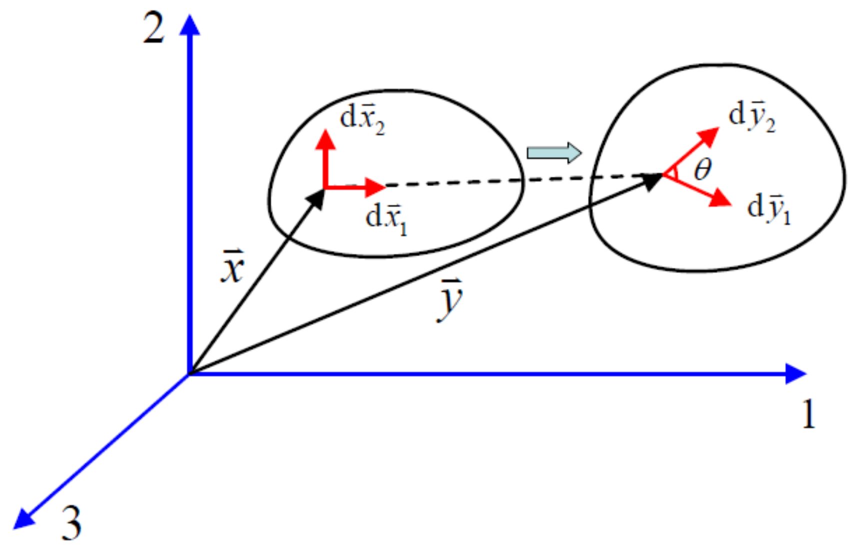
$$\mathrm{d}l_1^2 = \mathrm{d}\bar{y}_1 \cdot \mathrm{d}\bar{y}_1 = \underline{F}\mathrm{d}\bar{x}_1 \cdot \underline{F}\mathrm{d}\bar{x}_1 = \mathrm{d}l_{10}\bar{e}_1 \cdot \underline{C}\mathrm{d}l_{10}\bar{e}_1 = \mathrm{d}l_{10}^2 C_{11}$$

$$C_{11} = \frac{\mathrm{d}l_1^2}{\mathrm{d}l_{10}^2} = \lambda_1^2$$

- C_{11} is the stretch of a fiber initially aligned along e_1.

Physical/Geometric Interpretations of C

- Consider two initially perpendicular fibers



$$d\vec{x}_1 = dl_{10}\vec{e}_1, \quad d\vec{x}_2 = dl_{20}\vec{e}_2$$

$$d\vec{y}_1 = dl_1\vec{n}_1, \quad d\vec{y}_2 = dl_2\vec{n}_2$$

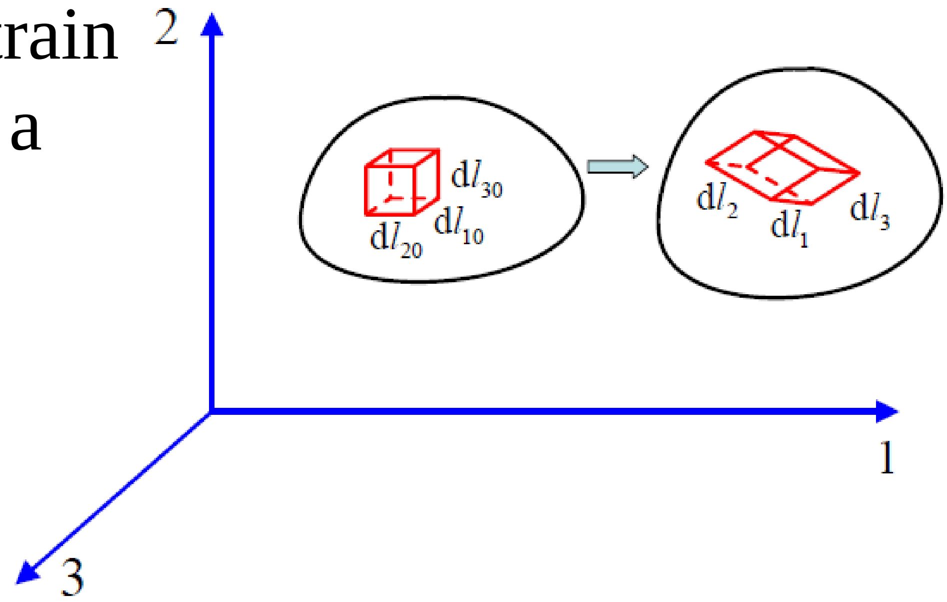
$$d\vec{y}_1 \cdot d\vec{y}_2 = \underline{F}d\vec{x}_1 \cdot \underline{F}d\vec{x}_2 = d\vec{x}_1 \cdot \underline{C}d\vec{x}_2 \Rightarrow dl_1 dl_2 \cos\theta_{12} = dl_{10}\vec{e}_1 \cdot \underline{C}dl_{20}\vec{e}_2$$

$$\cos\theta_{12} = \frac{C_{12}}{\lambda_1\lambda_2} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$$

- C_{12} is a measure of the angle between two fibers initially aligned in the e_1 and e_2 directions.

Physical/Geometric Interpretations of C

- The right Cauchy-Green strain gives the information how a small block of material deforms.



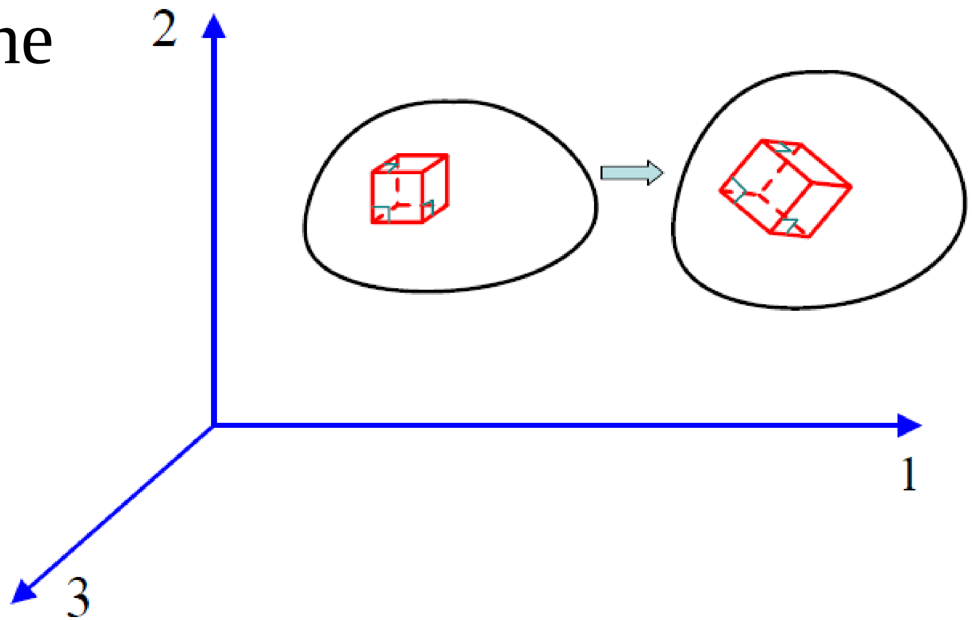
$$\underset{\sim}{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\mathrm{d}l_1 = \sqrt{C_{11}}\,\mathrm{d}l_{10}, \quad \mathrm{d}l_2 = \sqrt{C_{22}}\,\mathrm{d}l_{20}, \quad \mathrm{d}l_3 = \sqrt{C_{33}}\,\mathrm{d}l_{30}$$

$$\cos\theta_{12} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}, \quad \cos\theta_{23} = \frac{C_{23}}{\sqrt{C_{22}C_{33}}}, \quad \cos\theta_{13} = \frac{C_{13}}{\sqrt{C_{11}C_{33}}}$$

Physical/Geometric Interpretations of C

- In principal coordinates, the above geometric interpretations suggest



$$\underline{C} = \begin{bmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{bmatrix}$$

$$\underline{C}\vec{m} = \lambda^2 \vec{m} \quad \Rightarrow \quad \vec{m}_I \perp \vec{m}_{II} \perp \vec{m}_{III}$$

- This indicates that the fibers along the principal directions will remain perpendicular to each other after deformation.

Lagrangian vs. Eulerian Descriptions

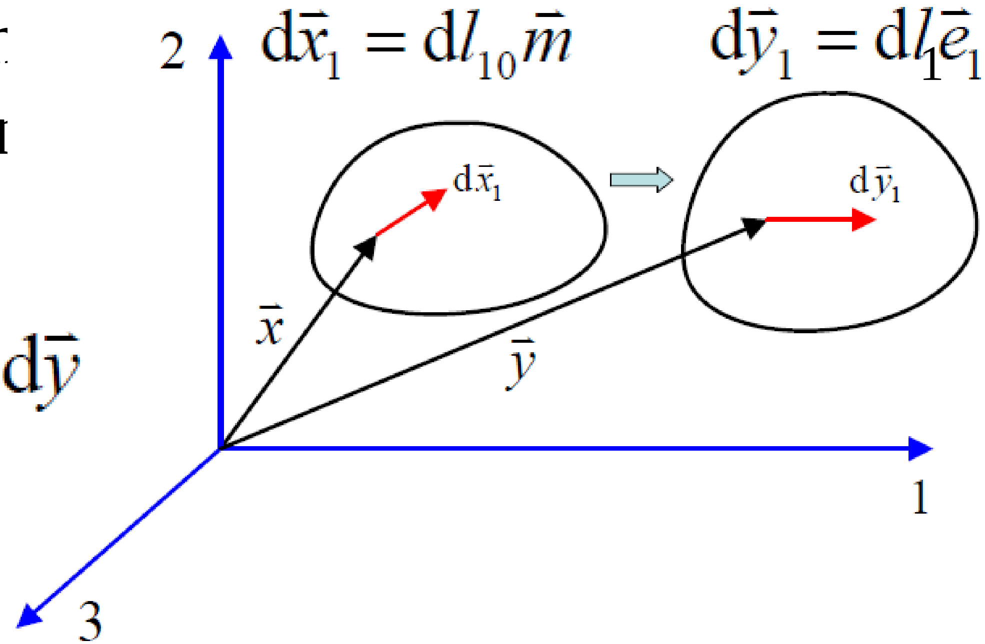
- The above formulation of strain has been focused on making predictions about the deformed configuration (called Eulerian) based on know information in the undeformed (called Lagrangian) configuration.
- Alternatively, we could reverse the direction of analysis. We could start from the deformed configuration and try to predict the undeformed configuration.

Joseph Louis Lagrange (1736-1813)

Leonard Euler (1707-1783)

Physical/Geometric Interpretations of B

- Consider a fiber aligned ir
  direction after deformatio



$$\mathrm{d}\vec{y} = \underline{F}\,\mathrm{d}\vec{x} \Rightarrow \mathrm{d}\vec{x} = \underline{F}^{-1}\mathrm{d}\vec{y}$$

$$\mathrm{d}\vec{x}_1 \cdot \mathrm{d}\vec{x}_1 = \mathrm{d}l_{10}^2 = F^{-1}\mathrm{d}\vec{y}_1 \cdot F^{-1}\mathrm{d}\vec{y}_1 = \mathrm{d}\vec{y}_1 \cdot \left(FF^T\right)^{-1}\mathrm{d}\vec{y}_1 = \mathrm{d}\vec{y}_1 \cdot \underline{B}^{-1}\mathrm{d}\vec{y}_1$$

$$\mathrm{d}l_{10}^2 = \mathrm{d}l_1\vec{e}_1 \cdot \underline{B}^{-1}\mathrm{d}l_1\vec{e}_1 \Rightarrow B_{11}^{-1} = \lambda_1^{-2}$$

- The stretch that has happened to a fiber aligned in the e_1 direction after deformation is given by B_{11}^{-1}

Physical/Geometric Interpretations of B

- The off-diagonal term has the following interpretation

$$\mathrm{d}\vec{x}_1 \cdot \mathrm{d}\vec{x}_2 = \mathrm{d}l_{10}\mathrm{d}l_{20}\cos\alpha_{12} = \mathrm{d}l_1\mathrm{d}l_2\vec{e}_1 \cdot \underline{B}^{-1}\vec{e}_2 \Rightarrow \cos\alpha_{12} = \frac{B_{12}^{-1}}{\sqrt{B_{11}^{-1}B_{22}^{-1}}}$$

- The original angle of two fibers that have become aligned in the e_1 and e_2 directions after deformation is given by B_{12}^{-1}

- In matrix form: $\underline{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, \quad \underline{B}^{-1} = \begin{bmatrix} B_{11}^{-1} & B_{12}^{-1} & B_{13}^{-1} \\ B_{21}^{-1} & B_{22}^{-1} & B_{23}^{-1} \\ B_{31}^{-1} & B_{32}^{-1} & B_{33}^{-1} \end{bmatrix}$

- In principal coordinates, we may define **the right stretch tensor**

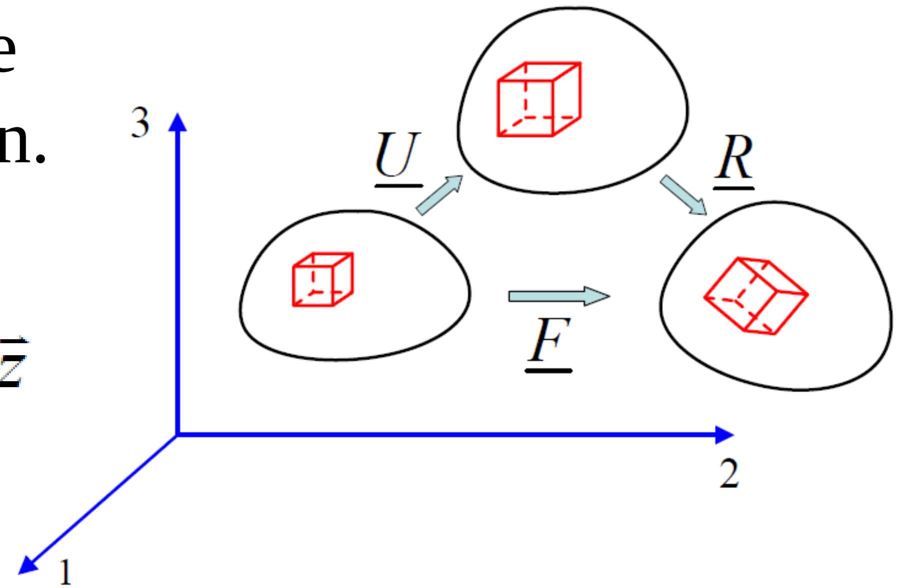
$$\underline{U} = \sqrt{\underline{C}} = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix}$$

Polar Decomposition

- A general deformation can be described as stretch + rotation.
- If stretch happens first

$$d\vec{x} \rightarrow d\vec{z} = \underline{U}\,d\vec{x} \rightarrow d\vec{y} = \underline{R}\,d\vec{z}$$

$$\Rightarrow d\vec{y} = \underline{R}\,\underline{U}\,d\vec{x} = \underline{F}\,d\vec{x}$$



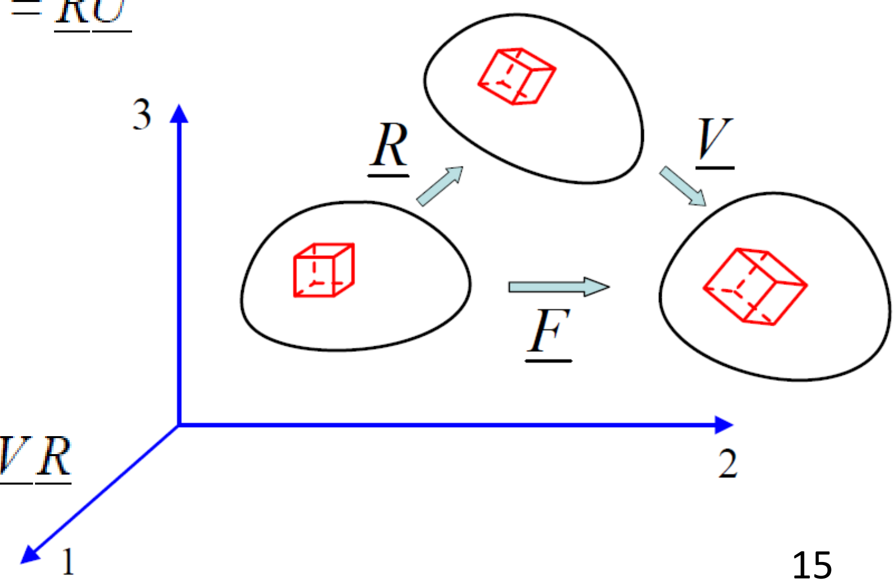
- Right polar decomposition: $\underline{F} = \underline{R}\,\underline{U}$

- If rotation happens first

$$d\vec{x} \rightarrow d\vec{z} = \underline{R}\,d\vec{x} \rightarrow d\vec{y} = \underline{V}\,d\vec{z}$$

$$\Rightarrow d\vec{y} = \underline{V}\,\underline{R}\,d\vec{x} = \underline{F}\,d\vec{x}$$

- Left polar decomposition: $\underline{F} = \underline{V}\,\underline{R}$

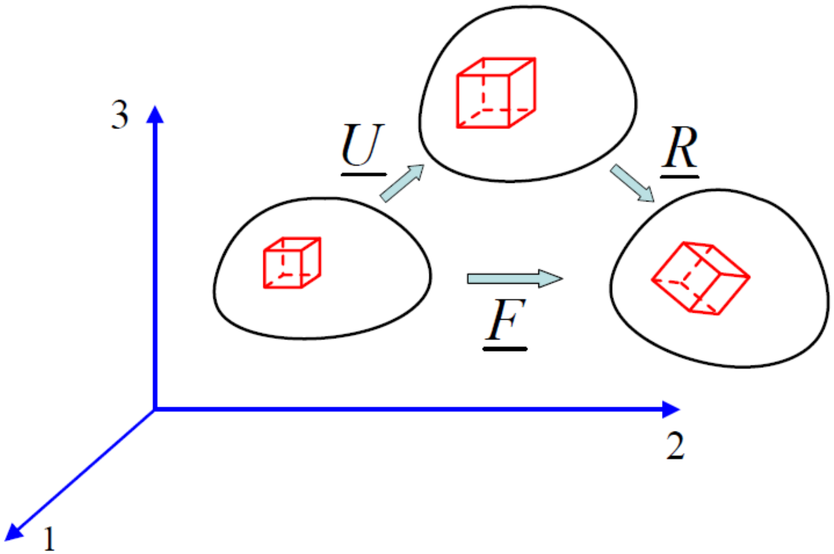


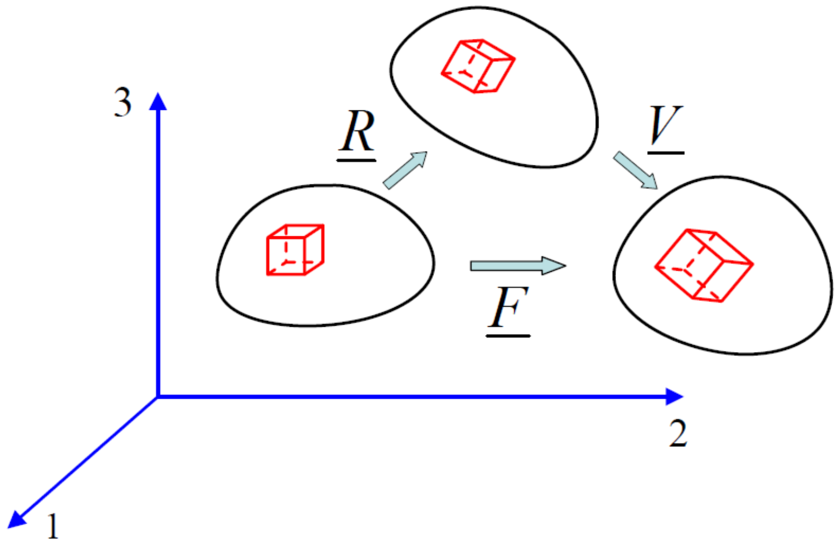
Polar Decomposition

- Cauchy-Green strain tensors and stretch tensors:

$$\underline{C} = \underline{F}^T \underline{F} = \left(\underline{R}\,\underline{U}\right)^T \underline{R}\,\underline{U} = \underline{U}^T \underline{R}^T \underline{R}\,\underline{U} = \underline{U}^T \underline{U} = \underline{U}^2$$

$$\underline{B} = \underline{F}\,\underline{F}^T = \underline{V}\,\underline{R}\left(\underline{V}\,\underline{R}\right)^T = \underline{V}\,\underline{R}\,\underline{R}^T \underline{V}^T = \underline{V}\,\underline{V}^T = \underline{V}^2$$





- One set of principal values and two sets of principal directions

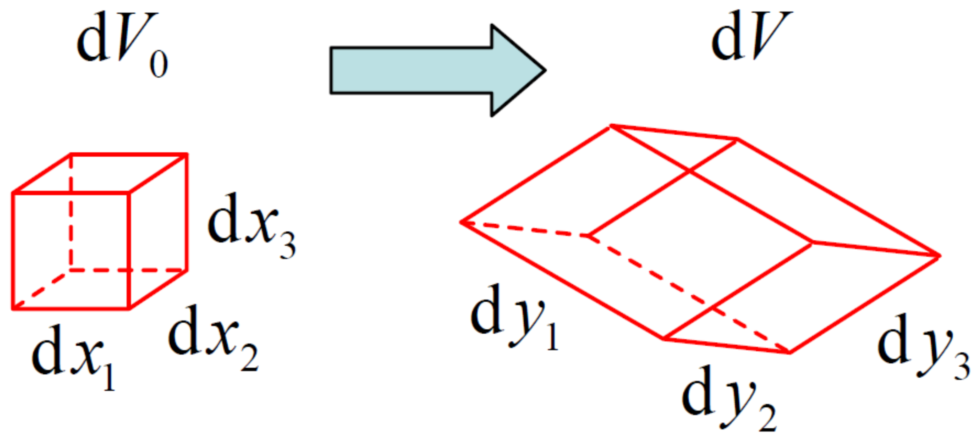
$$\underline{U}\vec{m} = \lambda_U \vec{m}, \quad \underline{C}\vec{m} = \lambda_U^2 \vec{m} \quad \Rightarrow \underline{R}\,\underline{U}\vec{m} = \lambda_U \underline{R}\vec{m} \quad \left(\underline{R}\,\underline{U} = \underline{F} = \underline{V}\,\underline{R}\right)$$

$$\Rightarrow \underline{V}\left(\underline{R}\vec{m}\right) = \lambda_U \left(\underline{R}\vec{m}\right) \quad \left(\underline{V}\vec{n} = \lambda_V \vec{n}, \underline{B}\vec{n} = \lambda_V^2 \vec{n}\right)$$

$$\Rightarrow \boxed{\lambda_U = \lambda_V, \quad \vec{n} = \underline{R}\vec{m}}$$

Jacobian of Deformation

- Volume after deformation:



$$
\begin{aligned}
\mathrm{d}V &= (\mathrm{d}y_1 \times \mathrm{d}y_2)\cdot \mathrm{d}y_3 = \varepsilon_{ijk}\mathrm{d}y_i\mathrm{d}y_j\mathrm{d}y_k \\
&= \varepsilon_{ijk}F_{ip}F_{jq}F_{kr}\mathrm{d}x_p\mathrm{d}x_q\mathrm{d}x_r \\
&= \varepsilon_{pqr}\det(\underline{F})\mathrm{d}x_p\mathrm{d}x_q\mathrm{d}x_r \\
&= \det(\underline{F})\mathrm{d}V_0
\end{aligned}
$$

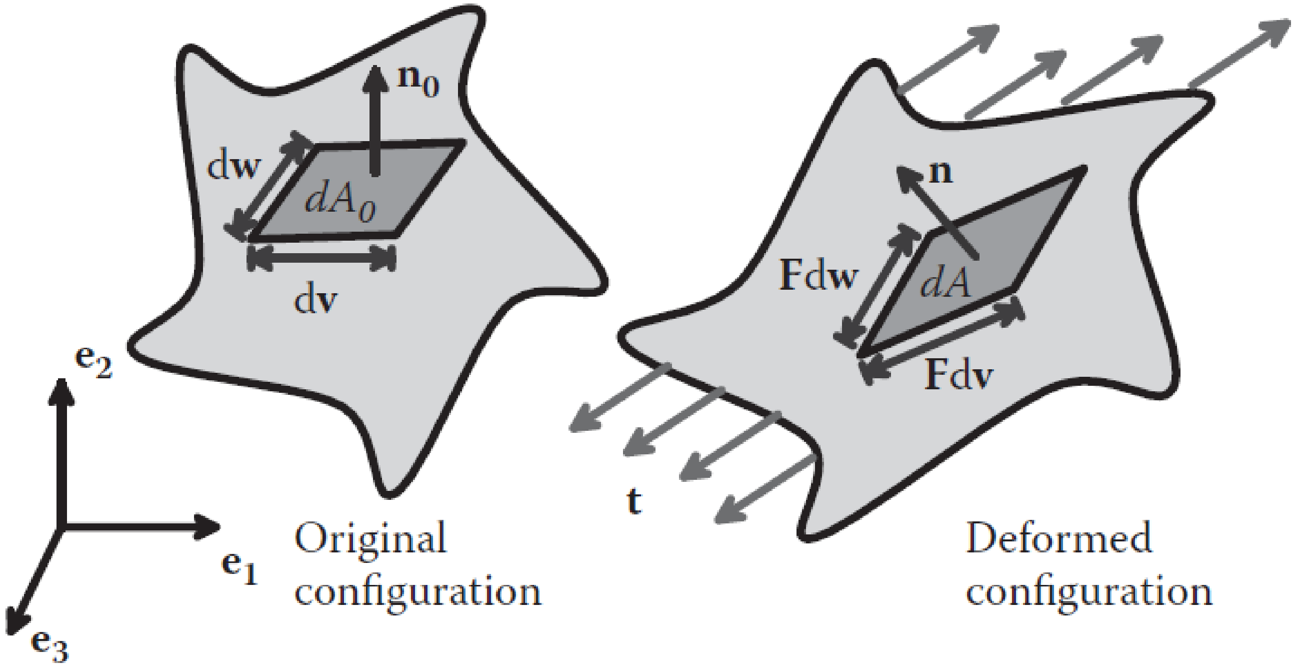
- The Jacobian of deformation (volume ratio):

$$
\frac{\mathrm{d}V}{\mathrm{d}V_0} = \det(\underline{F}) = \det(\underline{U}) = \det(\underline{V}) = \sqrt{\det(\underline{B})} = \sqrt{\det(\underline{C})} = J
$$

- The derivatives of **J** with respect to the components of **F** (can be proved by expanding **J** using index notation)

$$
\frac{\partial J}{\partial F_{ij}} = JF_{ji}^{-1}
$$

Deformation of an Area Element



$$dA_0 n_i^0 = \varepsilon_{ijk} dv_j dw_k \Rightarrow dA n_i = \varepsilon_{ijk} dv'_j dw'_k = \underline{\varepsilon_{ijk}} F_{jm} dv_m F_{kn} dw_n$$

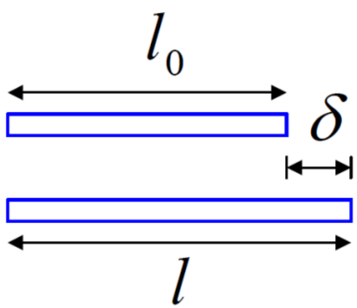
$$= \underline{\delta_{pi} \varepsilon_{pjk}} F_{jm} F_{kn} dv_m dw_n = \underline{F_{pl} F_{li}^{-1} \varepsilon_{pjk}} F_{jm} F_{kn} dv_m dw_n$$

$$= F_{li}^{-1} \underline{\varepsilon_{pjk} F_{pl} F_{jm} F_{kn}} dv_m dw_n = F_{li}^{-1} J \underline{\varepsilon_{lmn} dv_m dw_n} = F_{li}^{-1} J dA_0 n_l^0$$

$$\Rightarrow \boxed{\frac{dA}{dA_0} n_i = J n_l^0 F_{li}^{-1} \qquad \frac{dA}{dA_0} \mathbf{n} = J \mathbf{n}^0 \cdot \mathbf{F}^{-1}}$$

Measures of Strain

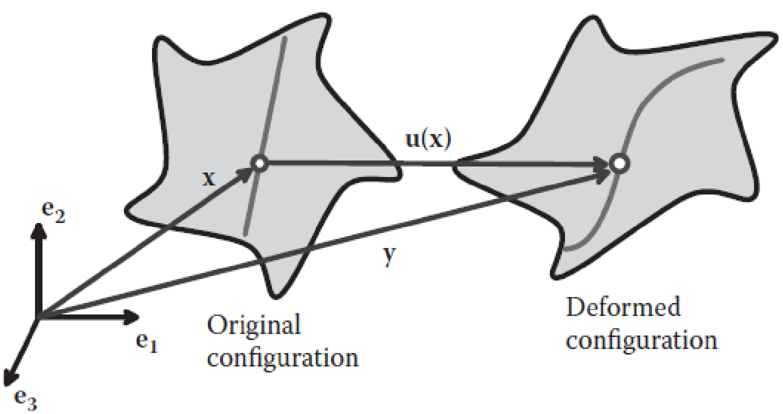
- Strains in 1D



$$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1 \qquad \varepsilon_{(3)} = \frac{l^2 - l_0^2}{2 l_0^2} = \frac{1}{2}\left(\lambda^2 - 1\right)$$

$$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l - l_0}{l} = 1 - \lambda^{-1} \qquad \varepsilon_{(4)} = \frac{l^2 - l_0^2}{2 l^2} = \frac{1}{2}\left(1 - \lambda^{-2}\right)$$

- Extension to 3D



$$(1):\ \left(\underline{U} - \underline{I}\right) \qquad (3):\ \frac{1}{2}\left(\underline{C} - \underline{I}\right)$$

$$(2):\ \left(\underline{I} - \underline{V}\right) \qquad (4):\ \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right)$$

$\underline{E} = \frac{1}{2}\left(\underline{C} - \underline{I}\right)$ is called the Lagrangian strain tensor.

$\underline{E}^* = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right)$ is called the Eulerian strain tensor.

Relation between Lagrangian and Eulerian Strain

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right), \quad E_{kl} = \frac{1}{2}\left(F_{mk}F_{ml} - \delta_{kl}\right)$$

$$\mathbf{E}^* = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}\right), \quad E^*_{mn} = \frac{1}{2}\left(\delta_{mn} - F^{-1}_{km}F^{-1}_{kn}\right)$$

$$\Rightarrow \begin{cases} \mathbf{F}^T \cdot \mathbf{E}^* \cdot \mathbf{F} = \frac{1}{2}\mathbf{F}^T \cdot \left(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}\right) \cdot \mathbf{F} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right) = \mathbf{E} \\ F_{mk}E^*_{mn}F_{nl} = \frac{1}{2}F_{mk}\left(\delta_{mn} - F^{-1}_{pm}F^{-1}_{pn}\right)F_{nl} = \frac{1}{2}\left(F_{mk}F_{ml} - \delta_{kl}\right) = E_{kl} \end{cases} \Rightarrow \boxed{\begin{array}{l} \mathbf{E} = \mathbf{F}^T \cdot \mathbf{E}^* \cdot \mathbf{F} \\ E_{kl} = F_{mk}E^*_{mn}F_{nl} \end{array}}$$

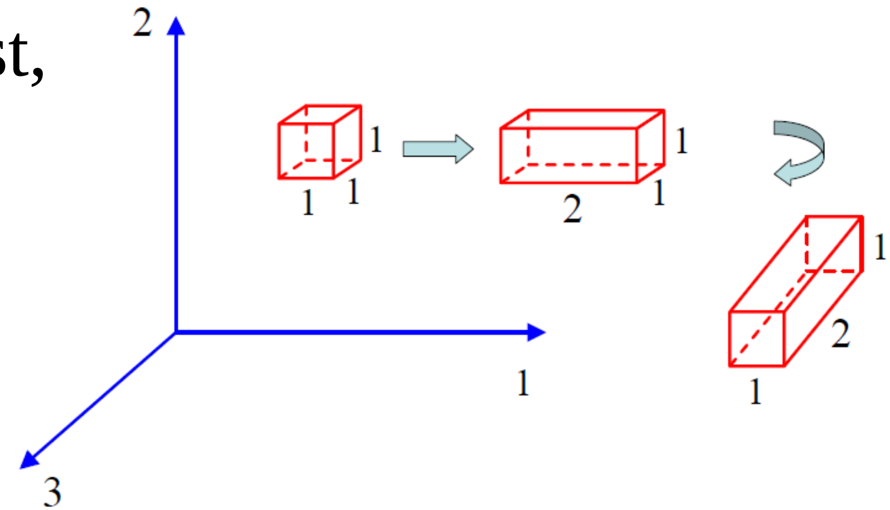
$$\Rightarrow \begin{cases} \mathbf{F}^{-T} \cdot \mathbf{E} \cdot \mathbf{F}^{-1} = \frac{1}{2}\mathbf{F}^{-T} \cdot \left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right) \cdot \mathbf{F}^{-1} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}\right) = \mathbf{E}^* \\ F^{-1}_{ki}E_{kl}F^{-1}_{lj} = F^{-1}_{ki}F_{mk}E^*_{mn}F_{nl}F^{-1}_{lj} = E^*_{ij} \end{cases} \Rightarrow \boxed{\begin{array}{l} \mathbf{E}^* = \mathbf{F}^{-T} \cdot \mathbf{E} \cdot \mathbf{F}^{-1} \\ E^*_{ij} = F^{-1}_{ki}E_{kl}F^{-1}_{lj} \end{array}}$$

Simple Deformations: Pure Dilation (no shear)

- Right stretch tensor: $\underline{U} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda \underline{I}$
- Right C-G strain: $\underline{C} = \lambda^2 \underline{I}$
- Deformation gradient: $\underline{F} = \underline{R}\underline{U} = \lambda \underline{R}$
- Left C-G strain: $\underline{B} = \underline{F}\,\underline{F}^T = \lambda^2 \underline{I}$
- Left stretch tensor: $\underline{V} = \lambda \underline{I}$
- Lagrangian and Eulerian strain: $\underline{E} = \frac{1}{2}\left(\lambda^2 - 1\right)\underline{I}, \underline{E}^* = \frac{1}{2}\left(1 - \lambda^{-2}\right)$
- Jacobian: $\frac{\mathrm{d}V}{\mathrm{d}V_0} = J = \det\left(\underline{F}\right) = \lambda^3$

Simple Deformations: Uniaxial Stretch

- Stretch along 1-direction first, then rotate 90^0 about 2-axis.



$$\lambda_I = \lambda\,,\quad \lambda_{II} = \lambda_{III} = 1$$

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R} = \vec{e}_i \cdot \underline{R}\,\vec{e}_j = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \underline{F} = \underline{R}\,\underline{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\underline{V}\,\underline{R} = \underline{F} \Rightarrow \underline{V} = \underline{F}\,\underline{R}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} == \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Simple Deformations: Uniaxial Stretch

$$\underline{C} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \underline{E}^* = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underline{E} = \frac{1}{2}\left(\underline{C} - \underline{I}\right) = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\mathrm{d}V}{\mathrm{d}V_0} = J = 2$$

- If rotated 45^0 about 2-axis:

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{R} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{F} = \underline{R}\,\underline{U}, \quad \underline{V}\,\underline{R} = \underline{R}\,\underline{U}$$

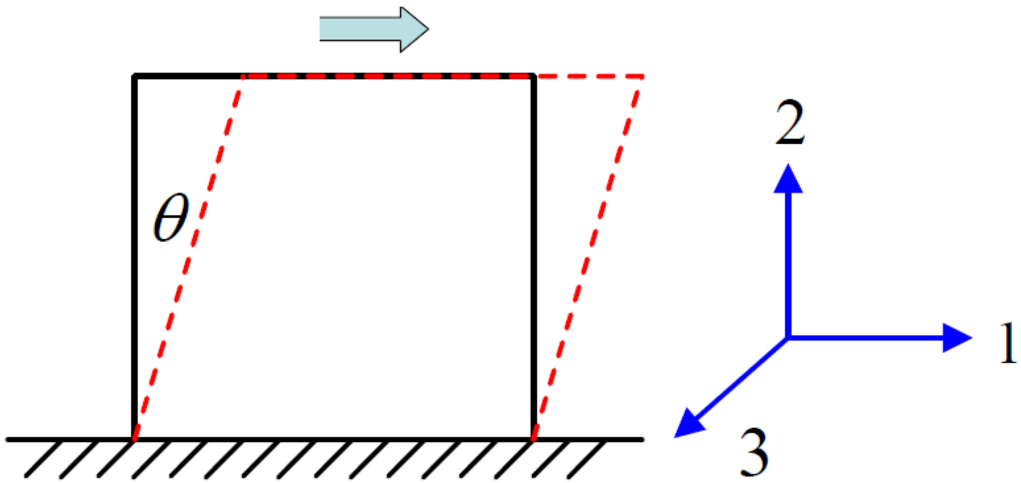
$$\underline{V} = \underline{F}\,\underline{R}^T, \quad \underline{B} = \underline{F}\,\underline{F}^T$$

Simple Deformations: Simple Shear

$$y_1 = x_1 + \tan\theta x_2$$

$$y_2 = x_2$$

$$y_3 = x_3$$



$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{F} = \underline{R}\,\underline{U}, \quad \underline{V}\,\underline{R} = \underline{R}\,\underline{U}$$

$$\underline{V} = \underline{F}\,\underline{R}^T, \quad \underline{B} = \underline{F}\,\underline{F}^T$$

$$\underline{C} = \underline{F}^T \underline{F} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta & 0 \\ \tan\theta & 1+\tan^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Small Strain & Small Rotation Theory

- Assumption: displacements in solids are usually small compared to relevant structure sizes: $$\frac{\partial u_i}{\partial x_j} \propto \left(\frac{u}{L}\right) << 1$$
- In this case, it proves to be useful to linearize all equations about the displacement gradient

$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right) = \delta_{ij} + u_{i,j} \quad (\text{because} \quad y_i = x_i + u_i\,)$$

$$\underline{C} = F_{ki}F_{kj} = \left(\delta_{ki} + u_{k,i}\right)\left(\delta_{kj} + u_{k,j}\right) = \delta_{ij} + u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{U} = \sqrt{\underline{C}} = \delta_{ij} + \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Small Strain & Small Rotation Theory

$$\underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij} \quad \text{(small strain tensor)}$$

$$\underline{B} = F_{ik}F_{jk} = \left(\delta_{ik} + u_{i,k}\right)\left(\delta_{jk} + u_{j,k}\right) \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{V} = \delta_{ij} + \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\underline{E}^{*} = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij} = \underline{E}$$

- For small strain & small rotation, there is no longer a need to distinguish between Lagrangian and Eulerian strain tensors. Basically, all strain tensors can be reduced to

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- This greatly simplifies the mathematical problem.

Small Strain & Small Rotation Theory

- For small strain and small rotation, the rigid body rotation part is analyzed as follows:

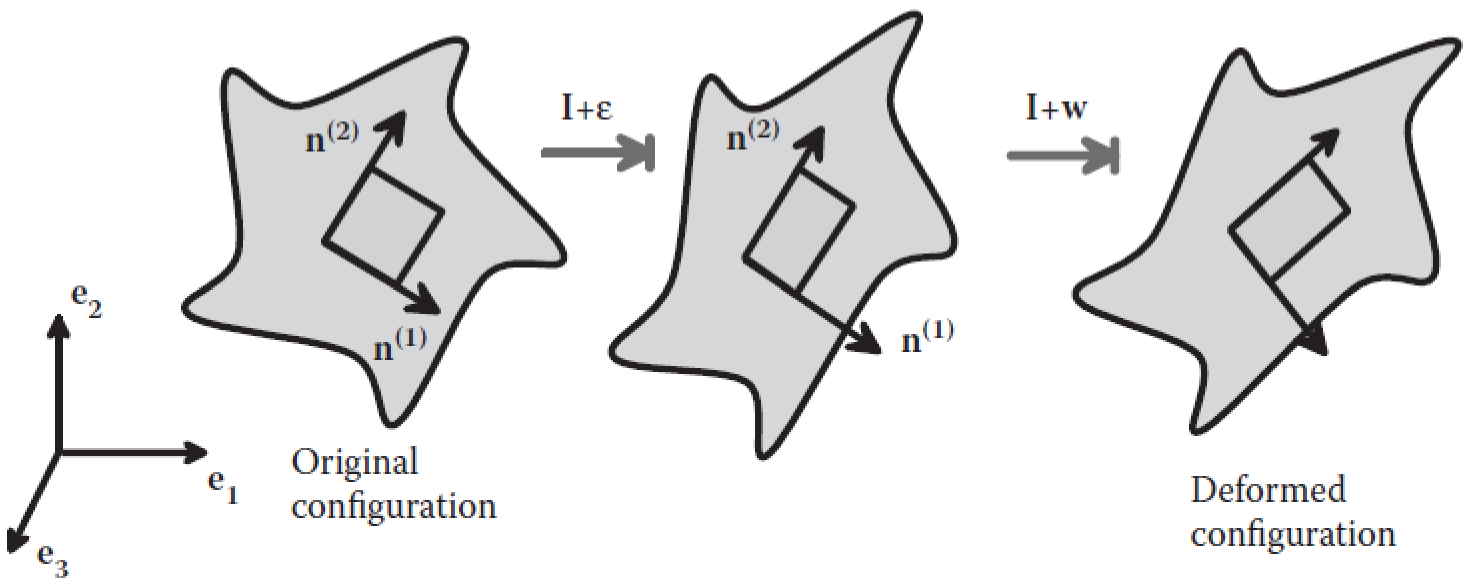
$$\underline{R} = \underline{I} + \underline{w}$$

$$\underline{F} = \underline{R}\,\underline{U} = \left(\underline{I} + \underline{w}\right)\left(\underline{I} + \underline{\varepsilon}\right) = \underline{I} + \underline{w} + \underline{\varepsilon},\ \underline{F} = \underline{I} + \vec{u}\nabla$$

$$\underline{w} = \vec{u}\nabla - \underline{\varepsilon} = \vec{u}\nabla - \frac{1}{2}\left(\vec{u}\nabla + \nabla\vec{u}\right) = \frac{1}{2}\left(\vec{u}\nabla - \nabla\vec{u}\right)$$

$$\boxed{\underline{w}^T = -\underline{w}, \quad w_{ji} = -w_{ij}}$$

- The small rotation tensor is anti-symmetric.



Strain Concepts and Their Generalizations in 3D

- Strain measures

1D	3D
$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1$	$\left(\underline{U} - \underline{I}\right)$
$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l - l_0}{l} = 1 - \lambda^{-1}$	$\left(\underline{I} - \underline{V}^{-1}\right)$
$\varepsilon_{(3)} = \frac{l^2 - l_0^2}{2 l_0^2} = \frac{1}{2}\left(\lambda^2 - 1\right)$	$\frac{1}{2}\left(\underline{C} - \underline{I}\right)$
$\varepsilon_{(4)} = \frac{l^2 - l_0^2}{2 l^2} = \frac{1}{2}\left(1 - \lambda^{-2}\right)$	$\frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right)$
For small strain: $\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon_{(3)} = \varepsilon_{(4)} = \varepsilon$	$\underline{\varepsilon} = \underline{U} - \underline{I} = \underline{I} - \underline{V}^{-1} = \frac{1}{2}\left(\underline{C} - \underline{I}\right) = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right)$ $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ or $\underline{\varepsilon} = \frac{1}{2}\left(\nabla\vec{u} + \nabla^T\vec{u}\right)$

Strain Concepts and Their Generalizations in 3D

- Strain-displacement relations

1 equation: $\varepsilon = \dfrac{\partial u}{\partial x}$	6 equations: $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \dfrac{\partial u_3}{\partial x_3}$ $\varepsilon_{12} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right),$ $\varepsilon_{23} = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right)$

Alternative Way of Small Strain Theory

- Normal strain

$$\varepsilon_{11} = \frac{\mathrm{d}l_1 - \mathrm{d}l_{10}}{\mathrm{d}l_{10}} = \frac{\sqrt{(\mathrm{d}l_{10} + \mathrm{d}u_1)^2 + \mathrm{d}u_2^2} - \mathrm{d}l_{10}}{\mathrm{d}l_{10}} = \sqrt{\left(1 + \frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2} - 1 \cong \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{22} = \frac{\mathrm{d}l_2 - \mathrm{d}l_{20}}{\mathrm{d}l_{20}} = \frac{\sqrt{(\mathrm{d}l_{20} + \mathrm{d}u_2)^2 + \mathrm{d}u_1^2} - \mathrm{d}l_{20}}{\mathrm{d}l_{20}} = \sqrt{\left(1 + \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2} - 1 \cong \frac{\partial u_2}{\partial x_2}$$

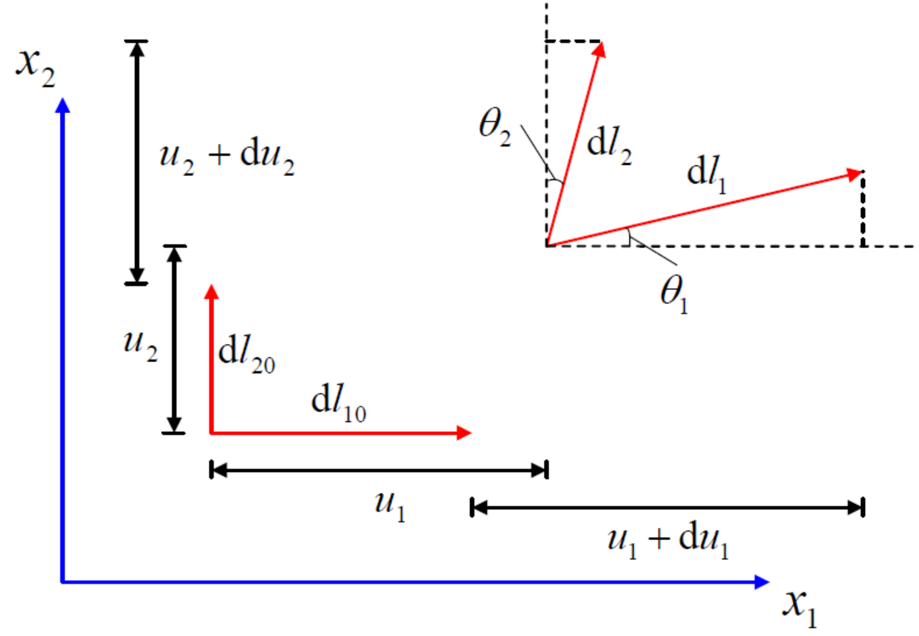
- Shear strain

$$\gamma_{12} = \theta_1 + \theta_2$$

$$\tan\theta_1 = \frac{\mathrm{d}u_2}{\mathrm{d}l_{10} + \mathrm{d}u_1} = \frac{\partial u_2 / \partial x_1}{1 + \partial u_1 / \partial x_1} = \frac{\partial u_2}{\partial x_1} = \theta_1$$

$$\tan\theta_2 = \frac{\mathrm{d}u_1}{\mathrm{d}l_{20} + \mathrm{d}u_2} = \frac{\partial u_1 / \partial x_2}{1 + \partial u_2 / \partial x_2} = \frac{\partial u_1}{\partial x_2} = \theta_2$$

$$\gamma_{12} = \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12}$$



Alternative Way of Small Strain Theory

- Volume strain

$$\frac{\mathrm{d}V}{\mathrm{d}V_0} = \det(\underline{F}) = \begin{vmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

If $\left|\frac{\partial u_i}{\partial x_j}\right| << 1$,

$$\frac{\mathrm{d}V}{\mathrm{d}V_0} = 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + o\left(\frac{\partial u_i}{\partial x_j}\right) \cong 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \varepsilon_{kk}$$

Therefore:

$$\frac{\mathrm{d}V - \mathrm{d}V_0}{\mathrm{d}V_0} = \varepsilon_{kk} \quad \text{(bulk strain)}$$

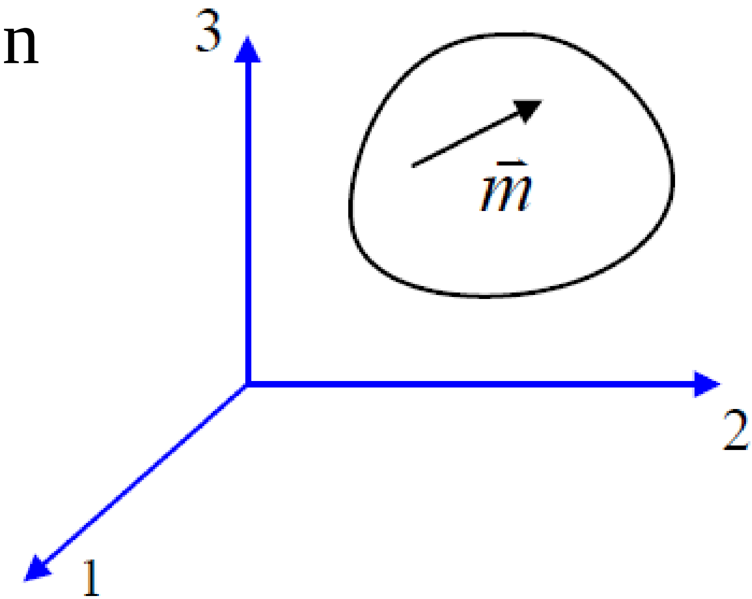
Strain Gauges

- Strain along an arbitrary direction

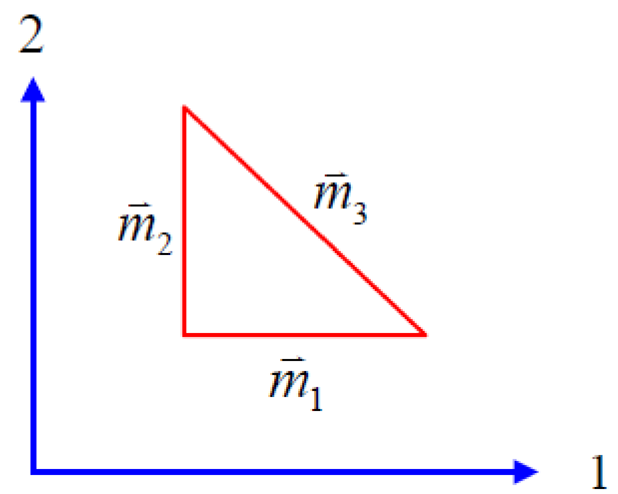
$$\vec{m} \cdot \underline{C} \cdot \vec{m} = \lambda_{\vec{m}}^2 \Rightarrow$$

$$\vec{m} \cdot \underline{\varepsilon} \cdot \vec{m} = \vec{m} \cdot \frac{1}{2}\left(\underline{C} - \underline{I}\right) \cdot \vec{m}$$

$$= \frac{1}{2}\left(\lambda_{\vec{m}}^2 - 1\right) \approx \lambda_{\vec{m}} - 1$$

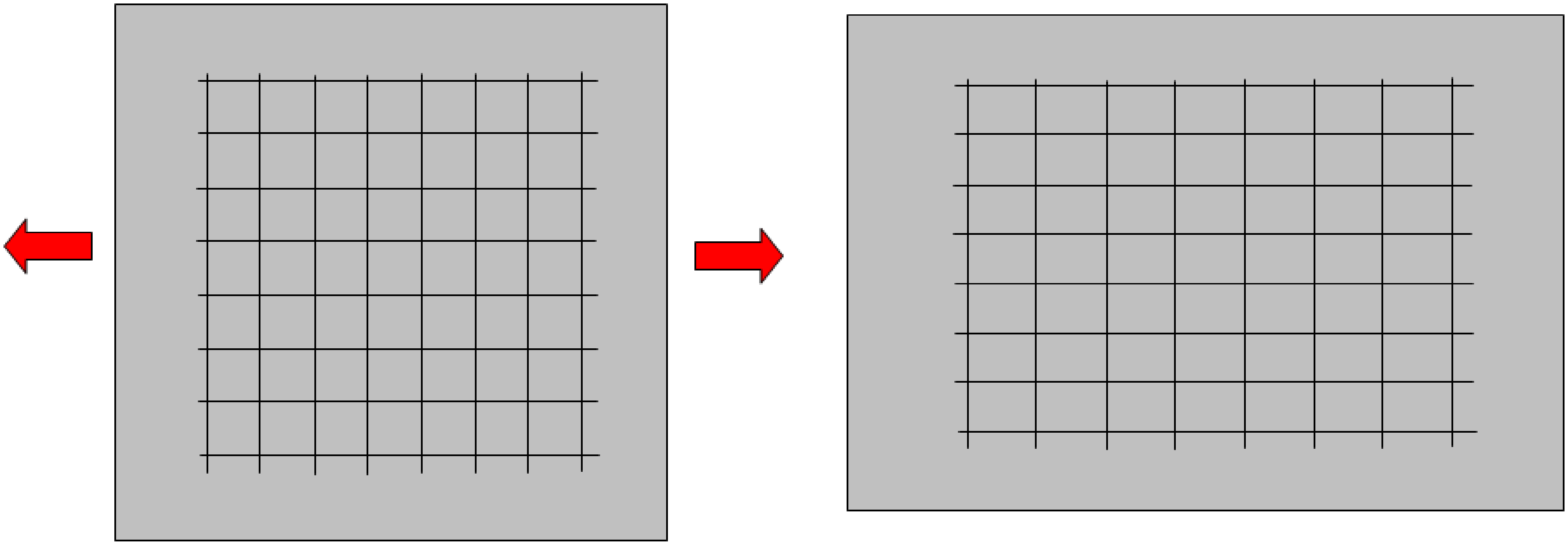


- Strain gauges aligned along different directions at a solid surface can be used to measure strain in the plane of the surface.



Strain Compatibility for Infinitesimal Strains

- Normally we want continuous and single-valued displacements; i.e. a mesh that fits perfectly together after deformation.



Undeformed State

Deformed State

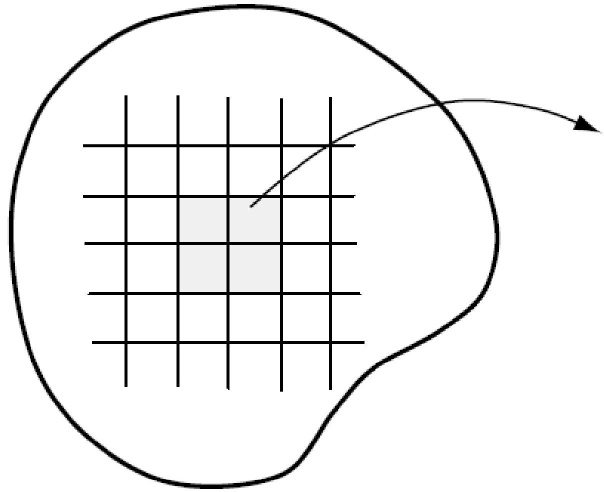
Strain Compatibility – Mathematical Context

- Strain-displacement relationship

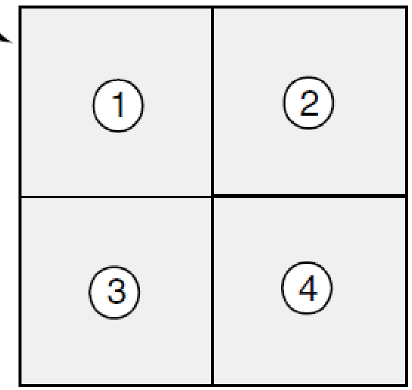
$$\varepsilon_x = \frac{\partial u}{\partial x}\,,\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z},\ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\ \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

- Given the three displacements: We have six equations to easily determine the six strain components.
- Given the six strains: We have six equations to determine three displacement components. This is an over-determined system and in general will not yield continuous single-valued displacements unless the strain components satisfy some additional relations.
- Not all symmetric second-order tensor fields can be strain fields.

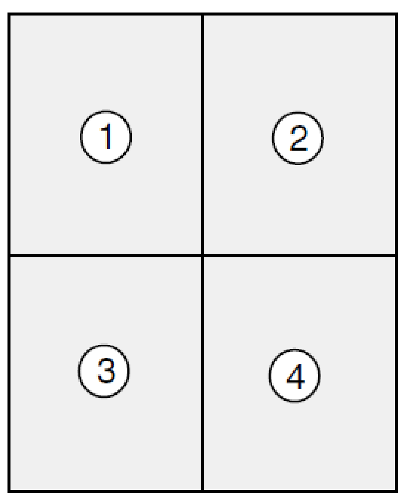
Strain Compatibility – Physical Interpretation



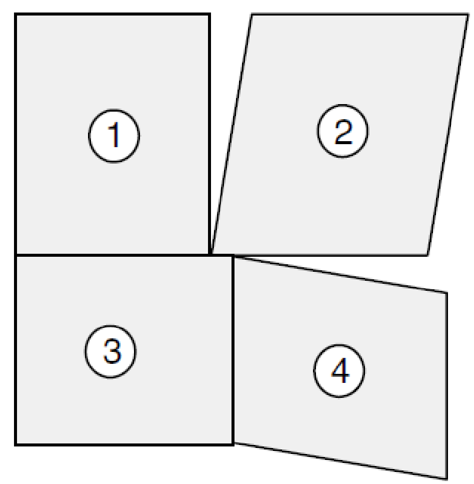
(a) Discretized Elastic Solid



(b) Undeformed Configuration



(c) Deformed Configuration
Continuous Displacements



(d) Deformed Configuration
Discontinuous Displacements

Compatibility Equations

- Differentiating twice the strain-displacement relationship

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \Rightarrow \begin{cases} \varepsilon_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right); \ \varepsilon_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right); \\ \varepsilon_{ik,jl} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right); \ \varepsilon_{jl,ik} = \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right). \end{cases}$$

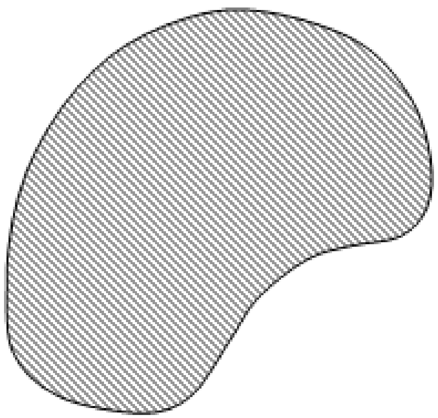
- The continuity of displacements implies the interchangeability of partial derivatives

$$\Rightarrow \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \qquad \in_{ipm}\in_{jqn} \frac{\partial^2 \varepsilon_{mn}}{\partial x_p \partial x_q} = 0$$

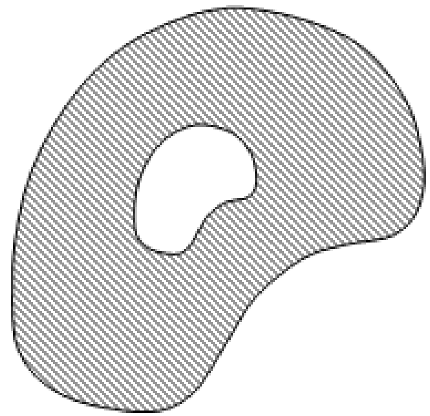
- This **strain compatibility condition** forms the **necessary and sufficient condition** for continuous and single-valuedness displacements (up to a rigid-body motion) in **simply connected regions**.
- For multiply connected regions, strain compatibility is necessary but no longer sufficient. Additional conditions must be imposed.

Domain Connectivity

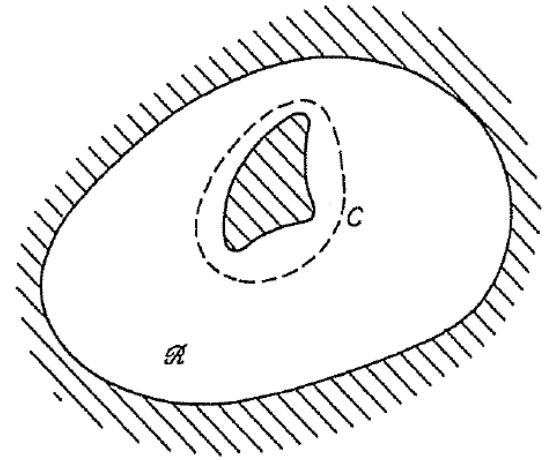
- Simply connected: all simple closed curves drawn in the region can be continuously shrunk to a point without going outside the region.

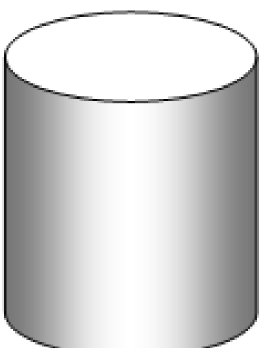


(a) Two-Dimensional
Simply Connected

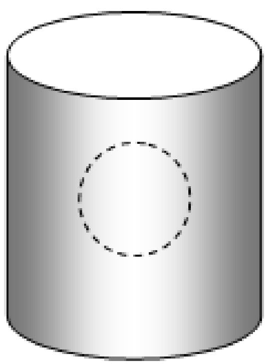


(b) Two-Dimensional
Multiply Connected

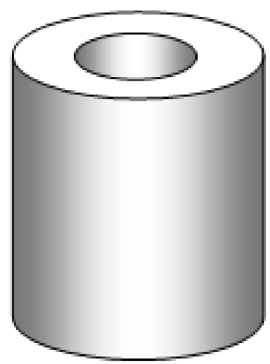




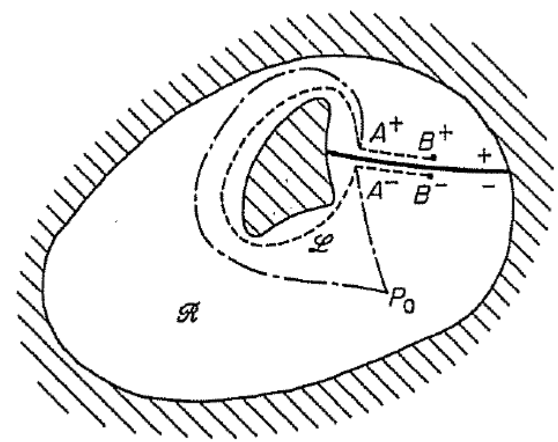
(c) Three-Dimensional
Simply Connected



(d) Three-Dimensional
Simply Connected



(e) Three-Dimensional
Multiply Connected



Compatibility Equations

- In 2-D, only 1 out of the 16 equations is meaningful and independent.

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}}$$

- Only 6 out of the 81 are meaningful

$$\boxed{1}: \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z};$$

$$\boxed{2}: \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z};$$

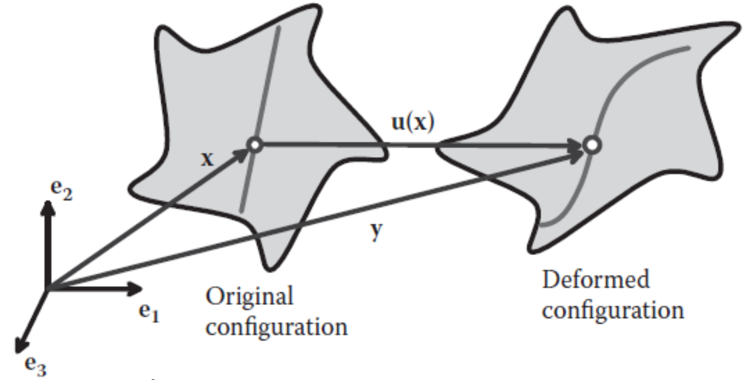
$$\boxed{3}: \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y};$$

$$\boxed{4}: \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right);$$

$$\boxed{5}: \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right);$$

$$\boxed{6}: \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right).$$

Material Time Derivative



- Displacement: $u_i(x_k,t) = y_i(x_k,t) - x_i$
- Velocity:

$$v_i(x_k,t) = \left.\frac{\partial y_i(x_k,t)}{\partial t}\right|_{x_k=\text{const}} = \left.\frac{\partial u_i(x_k,t)}{\partial t}\right|_{x_k=\text{const}}$$

- Acceleration: $a_i(x_k,t) = \dfrac{\partial v_i(x_k,t)}{\partial t} = \left.\dfrac{\partial^2 u_i(x_k,t)}{\partial t^2}\right|_{x_k=\text{const}}$

- With Eulerian description:

$$f = f(y_k,t) = f\left(y_k(x_m,t),t\right), \vec{v} = v_i(y_k,t) = v_i\left(y_k(x_m,t),t\right), \underline{\sigma} = \sigma_{ij}\left(y_k(x_m,t),t\right),$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y_k}\frac{\partial y_k}{\partial t} = \frac{\partial f}{\partial t} + v_k\frac{\partial f}{\partial y_k}$$

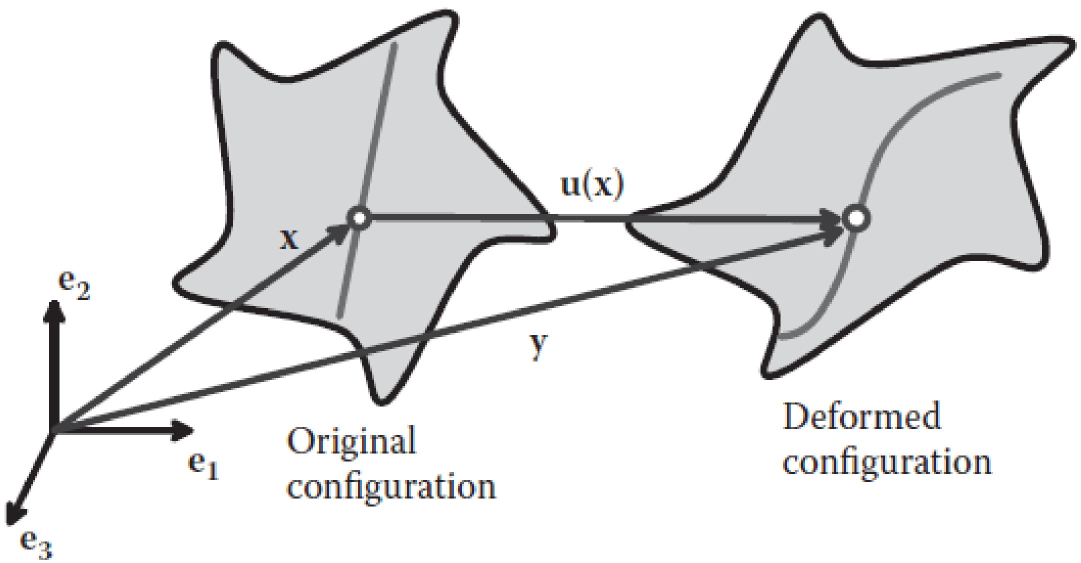
$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial y_k}\frac{\partial y_k}{\partial t} = \frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial y_k}$$

$$\frac{D\sigma_{ij}}{Dt} = \frac{\partial \sigma_{ij}}{\partial t} + \frac{\partial \sigma_{ij}}{\partial y_k}\frac{\partial y_k}{\partial t} = \frac{\partial \sigma_{ij}}{\partial t} + v_k\frac{\partial \sigma_{ij}}{\partial y_k}$$

Spatial time derivative

Rate of Deformation: Velocity Gradient

- Velocity gradient is the basic measure of deformation rate.
- Quantifies the relative velocities of two material particles.



$$v_i(x_k,t)=\frac{\partial y_i}{\partial t}=\left.\frac{\partial u_i(x_k,t)}{\partial t}\right|_{x_k=\text{const}}$$

$$\mathbf{L}=\mathbf{v}\otimes\nabla_{\mathrm{y}}\equiv L_{ij}=\frac{\partial v_i}{\partial y_j}$$

$$dv_i=v_i(\mathbf{y}+d\mathbf{y})-v_i(\mathbf{y})=\frac{\partial v_i}{\partial y_j}dy_j$$

$$dv_i=\frac{d}{dt}dy_i=\frac{d}{dt}\left(F_{ij}dx_j\right)=\dot{F}_{ij}dx_j \qquad dy_j=F_{ji}dx_i\Rightarrow dx_j=F_{jk}^{-1}dy_k$$

$$dv_i=\dot{F}_{ij}F_{jk}^{-1}dy_k$$

$$\boxed{\mathbf{v}\otimes\nabla_{\mathrm{y}}=\dot{\mathbf{F}}\cdot\mathbf{F}^{-1} \qquad \frac{\partial v_i}{\partial y_j}=\dot{F}_{ik}F_{kj}^{-1}}$$

Decomposition of Velocity Gradient

- Stretch rate: $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$
- Spin rate: $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$

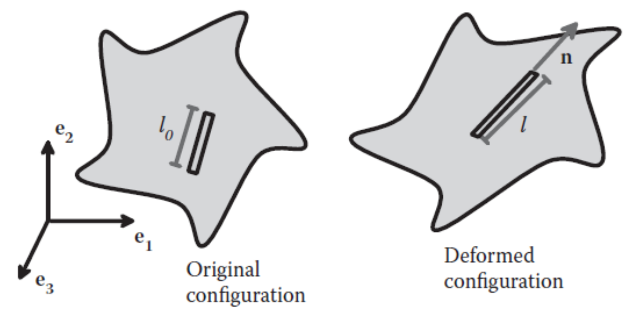
$$\mathbf{L} = \mathbf{D} + \mathbf{W} \quad L_{ij} = D_{ij} + W_{ij}$$

- **D** quantifies the rate of stretching of a material fiber in the deformed solid: $\dfrac{d}{dt}d\mathbf{y} = \dfrac{dl}{dt}\mathbf{n} + l\dfrac{d\mathbf{n}}{dt}$

$$\frac{d}{dt}d\mathbf{y} = \frac{d}{dt}(\mathbf{F}\cdot d\mathbf{x}) = \dot{\mathbf{F}}\cdot d\mathbf{x} = \dot{\mathbf{F}}\cdot\left(\mathbf{F}^{-1}d\mathbf{y}\right) = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1}\cdot d\mathbf{y} = \mathbf{L}\cdot d\mathbf{y} = (\mathbf{D}+\mathbf{W})\cdot l\mathbf{n}$$

$$(\mathbf{D}+\mathbf{W})\cdot l\mathbf{n} = \frac{dl}{dt}\mathbf{n} + l\frac{d\mathbf{n}}{dt}$$

$$\mathbf{n}\cdot d\mathbf{n}/dt = 0 \qquad \mathbf{n}\cdot\mathbf{W}\cdot\mathbf{n} = 0$$



$$\mathbf{n}\cdot(\mathbf{D}+\mathbf{W})\cdot l\mathbf{n} = \frac{dl}{dt}\mathbf{n}\cdot\mathbf{n} + l\mathbf{n}\cdot\frac{d\mathbf{n}}{dt} \Rightarrow \mathbf{n}\cdot\mathbf{D}\cdot l\mathbf{n} = \frac{dl}{dt}$$

- **W** provides a measure of the average angular velocity of all material fibers passing through a material point.

Infinitesimal Strain Rate and Rotation Rate

- For small strains, the rate of deformation (stretching) tensor can be approximated by the infinitesimal strain rate, whereas the spin can be approximated by the time derivative of the infinitesimal rotation tensor:

$$\frac{d}{dt}\boldsymbol{\varepsilon} = \frac{d}{dt}\frac{1}{2}\left(\mathbf{u}\otimes\nabla + (\mathbf{u}\otimes\nabla)^T\right) \approx \mathbf{D} \quad \text{or} \quad \dot{\varepsilon}_{ij} \approx D_{ij}$$

$$\frac{d}{dt}\mathbf{w} = \frac{d}{dt}\frac{1}{2}\left(\mathbf{u}\otimes\nabla - (\mathbf{u}\otimes\nabla)^T\right) \approx \mathbf{W} \quad \text{or} \quad \dot{w}_{ij} \approx W_{ij}$$

$$\frac{d}{dt}\frac{\partial u_i}{\partial x_j} = \dot{F}_{ij} = \dot{\varepsilon}_{ij} + \dot{w}_{ij} \approx L_{ij}$$

- The rate of deformation tensor can be related to time derivatives of other strain measures.