Thermoelasticity

Outline

- Heat Conduction Equation
- General 3-D Formulation
- Combined Plane Hooke's Law
- Stress Compatibility and Airy Stress Function
- Displacement Equilibrium and Displacement Potentials
- Thermal Stresses in Thin-Plates
- Summary of Solution Strategy
- Polar Coordinate: Airy Stress Function
- Polar Coordinate: Displacement Potentials
- Axi-symmetric Problems – Direct Solution
- Thermal Stresses in Circular Plates

Heat Conduction Equation

- Flow of heat in solids is associated with temperature differences
- For isotropic case, the **heat flux** is related to **temperature gradient** through **thermal conductivity**

$$q_i = -kT_{,i}$$

- From the principle of conservation of energy, **the uncoupled heat conduction equation** is given by

$$k\nabla^2 T = \rho c \frac{\partial T}{\partial t} - \rho h$$

ρ: mass density

c: specific heat capacity at constant volume

h: prescribed energy source term.

Heat Conduction Equation

- For zero heat sources and steady state, the heat onduction becomes Laplace equation

$$\nabla^2 T = 0.$$

- With appropriate thermal BCs, i.e. specified temperature or heat flux, the temperature field can be determined independent of the stress-field calculations.
- Once the temperature is obtained, elastic stress analysis procedures can then be employed to complete the problem solution.
- For us, the temperature distribution is usually a given condition.

General Formulation of Thermoelasticity – 3D

- Strain-displacement relations: $\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$
- Strain compatibility: $\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0$
- Equilibrium: $\sigma_{ij,j}+F_i=0$
- Thermoelastic Hooke's Law:

$$\varepsilon_{ij}=\frac{1}{2G}\sigma_{ij}-\frac{\nu}{2G(1+\nu)}\sigma_{kk}\delta_{ij}+\alpha T\delta_{ij},\quad \sigma_{ij}=\frac{2G\nu}{1-2\nu}\left(\varepsilon_{kk}-3\alpha T\right)\delta_{ij}+2G\left(\varepsilon_{ij}-\alpha T\delta_{ij}\right)$$

- Steady state heat conduction equation: $\nabla^2 T=0$
- 16 equations for 16 unknowns (3 displacements, 6 strains, 6 stresses and T):

$$f\{u_i,\varepsilon_{ij},\sigma_{ij};\lambda,G,F_i,T\}=0$$

- **3-D thermoelastic problems are way too difficult…**

Formulation of Thermoelasticity – 2D

- **Plane strain** thermoelastic Hooke's law

$$\varepsilon_{ij}=\frac{1}{2G}\sigma_{ij}-\frac{\nu}{2G(1+\nu)}\sigma_{kk}\delta_{ij}+\alpha T\delta_{ij}$$

$$0=\varepsilon_z \Rightarrow \sigma_z=\nu(\sigma_x+\sigma_y)-2G(1+\nu)\alpha T \Rightarrow \sigma_{kk}=(1+\nu)(\sigma_x+\sigma_y-2G\alpha T)$$

$$\Rightarrow \boxed{\begin{array}{l}\varepsilon_x=\dfrac{1}{2G}\left[(1-\nu)\sigma_x-\nu\sigma_y\right]+(1+\nu)\alpha T,\\ \varepsilon_y=\dfrac{1}{2G}\left[(1-\nu)\sigma_y-\nu\sigma_x\right]+(1+\nu)\alpha T,\ \varepsilon_{xy}=\dfrac{1}{2G}\sigma_{xy}\end{array}}$$

$$\sigma_{ij}=\frac{2G\nu}{1-2\nu}(\varepsilon_{kk}-3\alpha T)\delta_{ij}+2G(\varepsilon_{ij}-\alpha T\delta_{ij})$$

$$\Rightarrow \boxed{\begin{array}{l}\sigma_x=2G\left[\dfrac{1-\nu}{1-2\nu}\varepsilon_x+\dfrac{\nu}{1-2\nu}\varepsilon_y-\dfrac{1+\nu}{1-2\nu}\alpha T\right],\\ \sigma_y=2G\left[\dfrac{1-\nu}{1-2\nu}\varepsilon_y+\dfrac{\nu}{1-2\nu}\varepsilon_x-\dfrac{1+\nu}{1-2\nu}\alpha T\right],\ \tau_{xy}=2G\varepsilon_{xy}\end{array}}$$

Formulation of Thermoelasticity – 2D

- **Plane stress** thermoelastic Hooke's law

$$\sigma_{ij}=\frac{2G\nu}{1-2\nu}\left(\varepsilon_{kk}-3\alpha T\right)\delta_{ij}+2G\left(\varepsilon_{ij}-\alpha T\delta_{ij}\right)$$

$$0=\sigma_z \Rightarrow \varepsilon_z=-\frac{\nu}{1-\nu}\left(\varepsilon_x+\varepsilon_y\right)+\frac{1+\nu}{1-\nu}\alpha T \Rightarrow \varepsilon_{kk}=\frac{1-2\nu}{1-\nu}\left(\varepsilon_x+\varepsilon_y\right)+\frac{1+\nu}{1-\nu}\alpha T$$

$$\Rightarrow \begin{array}{l} \sigma_x=\frac{2G}{1-\nu}\left[\varepsilon_x+\nu\varepsilon_y-(1+\nu)\alpha T\right], \\ \sigma_y=\frac{2G}{1-\nu}\left[\varepsilon_y+\nu\varepsilon_x-(1+\nu)\alpha T\right],\ \tau_{xy}=2G\varepsilon_{xy} \end{array}$$

$$\varepsilon_{ij}=\frac{1}{2G}\sigma_{ij}-\frac{\nu}{2G(1+\nu)}\sigma_{kk}\delta_{ij}+\alpha T\delta_{ij}$$

$$\Rightarrow \begin{array}{l} \varepsilon_x=\frac{1}{2G}\left[\frac{1}{1+\nu}\sigma_x-\frac{\nu}{1+\nu}\sigma_y\right]+\alpha T, \\ \varepsilon_y=\frac{1}{2G}\left[\frac{1}{1+\nu}\sigma_y-\frac{\nu}{1+\nu}\sigma_x\right]+\alpha T,\ \varepsilon_{xy}=\frac{1}{2G}\tau_{xy} \end{array}$$

Formulation of Thermoelasticity – 2D

- **Combined plane** thermoelastic Hooke's law
- Define two material constants that are related to ν

$$\text{For plane strain: } \kappa = 3 - 4\nu \text{ or } \nu = \frac{3-\kappa}{4}, \ \eta = \nu;$$

$$\text{For plane stress: } \kappa = \frac{3-\nu}{1+\nu} \text{ or } \nu = \frac{3-\kappa}{1+\kappa}, \ \eta = 0.$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\sigma_{\alpha\beta} - \frac{3-\kappa}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right) + (1+\eta)\alpha T\delta_{\alpha\beta}, \quad \sigma_{\alpha\beta} = 2G\left[\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + \frac{2(1+\eta)}{1-\kappa}\alpha T\right]$$

$$\varepsilon_x = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_x - \frac{3-\kappa}{1+\kappa}\sigma_y\right) + (1+\eta)\alpha T,$$

$$\varepsilon_y = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_y - \frac{3-\kappa}{1+\kappa}\sigma_x\right) + (1+\eta)\alpha T, \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}.$$

$$\sigma_x = -\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_x + (3-\kappa)\varepsilon_y - 4(1+\eta)\alpha T\right],$$

$$\sigma_y = -\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_y + (3-\kappa)\varepsilon_x - 4(1+\eta)\alpha T\right], \tau_{xy} = 2G\varepsilon_{xy}.$$

Stress Formulation – 2D

- Beltrami-Michell Equation:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=2\frac{\partial^2 \varepsilon_{xy}}{\partial x\partial y}$$

$$\boxed{\varepsilon_x=\frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_x-\frac{3-\kappa}{1+\kappa}\sigma_y\right)+(1+\eta)\alpha T,\varepsilon_y=\frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_y-\frac{3-\kappa}{1+\kappa}\sigma_x\right)+(1+\eta)\alpha T,\varepsilon_{xy}=\frac{1}{2G}\sigma_{xy}.}$$

$$\Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{1+\kappa}{4}\sigma_x-\frac{3-\kappa}{4}\sigma_y+2G(1+\eta)\alpha T\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1+\kappa}{4}\sigma_y-\frac{3-\kappa}{4}\sigma_x+2G(1+\eta)\alpha T\right)=2\frac{\partial^2 \sigma_{xy}}{\partial x\partial y}$$

Add $\left(\frac{\partial^2 \sigma_x}{\partial x^2}+\frac{\partial^2 \sigma_y}{\partial y^2}\right)$ to both sides: $\frac{1+\kappa}{4}\nabla^2\left(\sigma_x+\sigma_y\right)+2G(1+\eta)\alpha\nabla^2 T=\frac{\partial^2 \sigma_x}{\partial x^2}+\frac{\partial^2 \sigma_y}{\partial y^2}+2\frac{\partial^2 \sigma_{xy}}{\partial x\partial y}$

Using Equilibrium on the RHS: $\frac{1+\kappa}{4}\nabla^2\left(\sigma_x+\sigma_y\right)+2G(1+\eta)\alpha\nabla^2 T=-\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$

$$\Rightarrow \boxed{\nabla^2\left(\sigma_x+\sigma_y\right)+\frac{8G(1+\eta)}{1+\kappa}\alpha\nabla^2 T=-\frac{4}{1+\kappa}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)}$$

For plane strain: $\kappa=3-4\nu$ or $\nu=\frac{3-\kappa}{4}$, $\eta=\nu$	For plane stress: $\kappa=\frac{3-\nu}{1+\nu}$ or $\nu=\frac{3-\kappa}{1+\kappa}$, $\eta=0$
$\nabla^2\left(\sigma_x+\sigma_y\right)+\frac{E}{1-\nu}\alpha\nabla^2 T=-\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$	$\nabla^2\left(\sigma_x+\sigma_y\right)+E\alpha\nabla^2 T=-(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$

Stress Function Formulation without Body Forces

- Air Stress Function Solution

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

- where $\psi = \psi(x,y)$ is an arbitrary form called *Airy's stress function*. This stress form automatically satisfies the equilibrium equation.

- Beltrami-Michell Equation:

$$\nabla^4 \psi + \frac{8G(1+\eta)}{1+\kappa} \alpha \nabla^2 T = 0$$

For plane strain: $\kappa = 3 - 4\nu, \ \eta = \nu$

$$\nabla^4 \psi + \frac{E}{1-\nu} \alpha \nabla^2 T = 0$$

For plane stress: $\kappa = \frac{3-\nu}{1+\nu}, \ \eta = 0$

$$\nabla^4 \psi + E \alpha \nabla^2 T = 0$$

Stress Function Formulation without Body Forces

- Solution of the Airy Stress Function

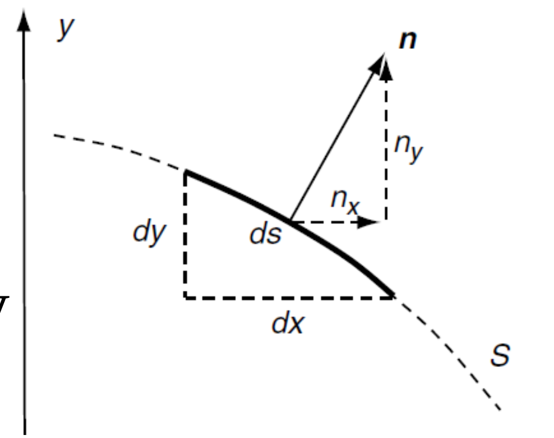
$$\nabla^4\psi+\frac{8G(1+\eta)}{1+\kappa}\alpha\nabla^2T=0$$

$$\psi=\psi^{(h)}+\psi^{(p)} \;\Rightarrow\; \nabla^4\psi^{(h)}=0,\; \nabla^2\psi^{(p)}+\frac{8G(1+\eta)}{1+\kappa}\alpha T=0$$

- The traction BCs in terms of Airy Stress Function

$$T_x^n=\sigma_x n_x+\tau_{xy}n_y=\frac{\partial^2\psi}{\partial y^2}\frac{\mathrm{d}y}{\mathrm{d}s}+\left(-\frac{\partial^2\psi}{\partial x\partial y}\right)\left(-\frac{\mathrm{d}x}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\psi}{\partial y}\right)$$

$$T_y^n=\tau_{xy}n_x+\sigma_y n_y=\left(-\frac{\partial^2\psi}{\partial x\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial^2\psi}{\partial x^2}\left(-\frac{\mathrm{d}x}{\mathrm{d}s}\right)=-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\psi}{\partial x}\right)$$



$$n_x=\frac{\mathrm{d}x}{\mathrm{d}n}=\frac{\mathrm{d}y}{\mathrm{d}s},\; n_y=\frac{\mathrm{d}y}{\mathrm{d}n}=-\frac{\mathrm{d}x}{\mathrm{d}s}$$

- Integrate over a portion of the boundary

$$\int_C T_x^n\mathrm{d}s+C_1=\frac{\partial\psi}{\partial y},\quad \int_C T_y^n\mathrm{d}s+C_2=-\frac{\partial\psi}{\partial x}$$

Stress Function Formulation without Body Forces

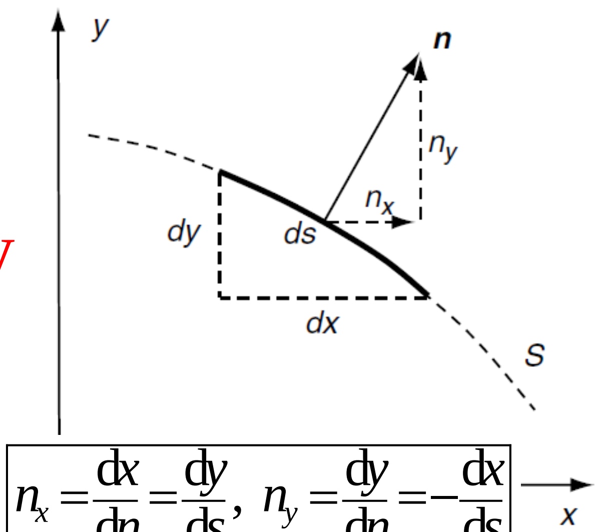
- Consider the directional derivative of the Airy Stress Function along the boundary normal

$$\Rightarrow \frac{d\psi}{dn} = \nabla\psi \cdot \boldsymbol{n} = \frac{\partial\psi}{\partial x} n_x + \frac{\partial\psi}{\partial y} n_y = \left(-\int_C T_y^n ds\right)\left(\frac{dy}{ds}\right) + \left(\int_C T_x^n ds\right)\left(-\frac{dx}{ds}\right) = -\boldsymbol{t}\cdot\boldsymbol{F}$$

- where $\boldsymbol{t}$ is the unit tangent vector and $\boldsymbol{F}$ is the resultant boundary force.
- For many applications, the BCs are simply expressed in terms of stresses.
- For the case of zero surface tractions:

$$d\psi/dn = 0 \quad \Rightarrow \quad \psi = C.$$

- For simply connected regions, a steady temperature distribution with zero boundary tractions will not affect the in-plane stress field.



$$n_x = \frac{dx}{dn} = \frac{dy}{ds}, \quad n_y = \frac{dy}{dn} = -\frac{dx}{ds}$$

Displacement Formulation – 2D

- Navier's equations

$$\sigma_x = -\frac{G}{1-\kappa}\left[(1+\kappa)\frac{\partial u}{\partial x} + (3-\kappa)\frac{\partial v}{\partial y} - 4(1+\eta)\alpha T\right],$$

$$\sigma_y = -\frac{G}{1-\kappa}\left[(1+\kappa)\frac{\partial v}{\partial y} + (3-\kappa)\frac{\partial u}{\partial x} - 4(1+\eta)\alpha T\right], \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

$$G\nabla^2 u - \frac{2G}{1-\kappa}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{4G(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial x} + F_x = 0, G\nabla^2 v - \frac{2G}{1-\kappa}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{4G(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial y} + F_y = 0$$

For plane strain: $\kappa = 3 - 4\nu$ or $\nu = \frac{3-\kappa}{4}$, $\eta = \nu$

$$G\nabla^2 u + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E}{1-2\nu}\alpha\frac{\partial T}{\partial x} + F_x = 0$$

$$G\nabla^2 v + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E}{1-2\nu}\alpha\frac{\partial T}{\partial y} + F_y = 0$$

For plane stress: $\kappa = \frac{3-\nu}{1+\nu}$ or $\nu = \frac{3-\kappa}{1+\kappa}$, $\eta = 0$

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E}{1-\nu}\alpha\frac{\partial T}{\partial x} + F_x = 0$$

$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{E}{1-\nu}\alpha\frac{\partial T}{\partial y} + F_y = 0$$

Displacement Formulation – 2D

- Stress/Traction Boundary Conditions

$$\begin{pmatrix} T_x^n = T_x^{(b)}(x,y) = \sigma_x^{(b)} n_x + \tau_{xy}^{(b)} n_y \\ T_y^n = T_y^{(b)}(x,y) = \tau_{xy}^{(b)} n_x + \sigma_y^{(b)} n_y \end{pmatrix} \text{ on } S_t$$

$$\Rightarrow \begin{cases} T_x - \dfrac{4G(1+\eta)}{1-\kappa}\alpha T n_x = -\dfrac{G}{1-\kappa}\left[(1+\kappa)\dfrac{\partial u}{\partial x} + (3-\kappa)\dfrac{\partial v}{\partial y}\right] n_x + G\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) n_y \\ T_y - \dfrac{4G(1+\eta)}{1-\kappa}\alpha T n_y = G\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) n_x - \dfrac{G}{1-\kappa}\left[(1+\kappa)\dfrac{\partial v}{\partial y} + (3-\kappa)\dfrac{\partial u}{\partial x}\right] n_y \end{cases}$$

- Displacement Boundary Conditions

$$u = u_b(x,y)\,,\; v = v_b(x,y) \text{ on } S_u$$

General Observation of Displacement Formulation

- From the displacement formulation, the solution to a 2-D thermoelastic problem can be superposed by the original isothermal solution and
- (a) additional body forces:

$$F_x = \frac{4G(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial x}, \quad F_y = \frac{4G(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial y},$$

- (b) additional normal surface traction:

$$T^n = -\frac{4G(1+\eta)}{1-\kappa}\alpha T.$$

- Displacements are derived for the additional equivalent body and surface forces. The stresses are:

$$\sigma_x = -\frac{G}{1-\kappa}\left[(1+\kappa)\frac{\partial u}{\partial x} + (3-\kappa)\frac{\partial v}{\partial y} - 4(1+\eta)\alpha T\right],$$

$$\sigma_y = -\frac{G}{1-\kappa}\left[(1+\kappa)\frac{\partial v}{\partial y} + (3-\kappa)\frac{\partial u}{\partial x} - 4(1+\eta)\alpha T\right], \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Displacement Potential Formulation – 2D

- Navier's (governing) equations without body forces

$$\nabla^2 u - \frac{2}{1-\kappa}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{4(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial x}=0,\quad \nabla^2 v - \frac{2}{1-\kappa}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{4(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial y}=0$$

- Assuming that the displacement vector is derivable from a scalar potential, i.e. a 2-D Lamé Strain Potential

$$\boldsymbol{u}=\nabla\phi:\ \left(u=\frac{\partial\phi}{\partial x},\ v=\frac{\partial\phi}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\nabla^2\phi\right)=\frac{4(1+\eta)}{1+\kappa}\alpha\frac{\partial T}{\partial x},\quad \frac{\partial}{\partial y}\left(\nabla^2\phi\right)=\frac{4(1+\eta)}{1+\kappa}\alpha\frac{\partial T}{\partial y}$$

- Integrating and dropping the constants of integration

$$\nabla^2\phi=\frac{4(1+\eta)}{1+\kappa}\alpha T$$

For plane strain: $\kappa=3-4\nu,\ \eta=\nu:\quad \nabla^2\phi=\frac{1+\nu}{1-\nu}\alpha T$

For plane stress: $\kappa=\frac{3-\nu}{1+\nu},\ \eta=0:\quad \nabla^2\phi=(1+\nu)\alpha T$

Displacement Potential Formulation – 2D

- Solution of the displacement potential

$$\nabla^2\phi = \frac{4(1+\eta)}{1+\kappa}\alpha T$$

$$\phi = \phi^{(h)} + \phi^{(p)} \;\Rightarrow\; \nabla^2\phi^{(h)} = 0,\; \nabla^2\phi^{(p)} = \frac{4(1+\eta)}{1+\kappa}\alpha T$$

- From the standard potential theory

$$\phi^{(p)} = \frac{1}{2\pi}\frac{4(1+\eta)}{1+\kappa}\alpha\iint_R T(\xi,\eta)\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\,\mathrm{d}\xi\mathrm{d}\eta.$$

$$\Rightarrow\; u^{(p)} = \frac{\partial\phi^{(p)}}{\partial x},\; v = \frac{\partial\phi^{(p)}}{\partial y}$$

- The total displacements

$$u = u^{(h)} + u^{(p)} = \frac{\partial}{\partial x}\left(\phi^{(h)} + \phi^{(p)}\right),\;\; v = v^{(h)} + v^{(p)} = \frac{\partial}{\partial y}\left(\phi^{(h)} + \phi^{(p)}\right)$$

- Next, we try to solve for the homogeneous displacement.

Displacement Potential Formulation – 2D

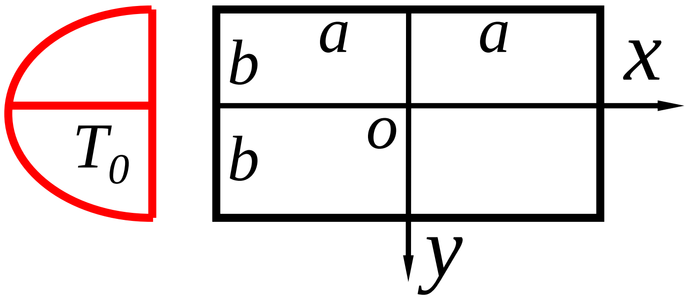
- The homogeneous solution satisfies the following isothermal Navier's equation

$$\nabla^2 u^{(h)} - \frac{2}{1-\kappa}\frac{\partial}{\partial x}\left(\frac{\partial u^{(h)}}{\partial x} + \frac{\partial v^{(h)}}{\partial y}\right) = 0, \qquad \nabla^2 v^{(h)} - \frac{2}{1-\kappa}\frac{\partial}{\partial y}\left(\frac{\partial u^{(h)}}{\partial x} + \frac{\partial v^{(h)}}{\partial y}\right) = 0$$

- The BCs for the homogeneous displacement = **the original conditions by subtracting the contributions of the particular displacement**

- Thus, with the particular solution known, the general problem is then reduced to solving an isothermal case.

Thermal Stresses in an Elastic Thin-Plate



- Given temperature change:

$$T = T_0\left(1-\frac{y^2}{b^2}\right)$$

- Navier's equation for plane stress

$$\nabla^2\phi = (1+\nu)\alpha T = (1+\nu)\alpha T_0\left(1-\frac{y^2}{b^2}\right)$$

- Since the temperature load is independent of *x*, let's try the particular displacement potential

$$\phi = Ay^2 + By^4 \;\Rightarrow\; \nabla^2\phi = 2A + 12By^2 = (1+\nu)\alpha T_0\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow A = \frac{(1+\nu)\alpha T_0}{2}, \qquad B = -\frac{(1+\nu)\alpha T_0}{12b^2}$$

$$\Rightarrow \phi = Ay^2 + By^4 = (1+\nu)\alpha T_0\left(\frac{y^2}{2} - \frac{y^4}{12b^2}\right)$$

Thermal Stresses in an Elastic Thin-Plate

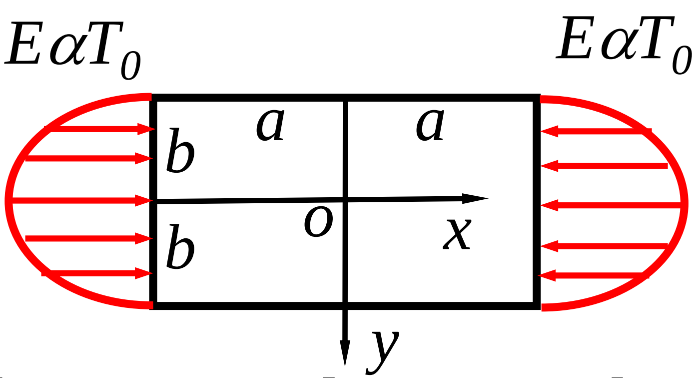
- Stresses due to the particular displacement potential

$$\Rightarrow u=\frac{\partial\phi}{\partial x}=0,\ v=\frac{\partial\phi}{\partial y}=(1+\nu)\alpha T_0\left(y-\frac{y^3}{3b^2}\right)$$

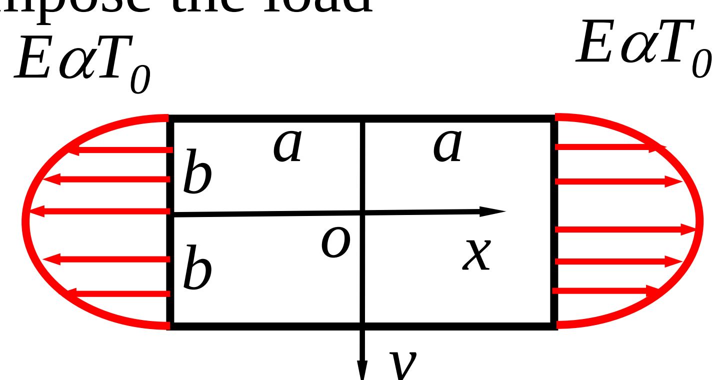
$$\Rightarrow \varepsilon_x=\frac{\partial u}{\partial x}=\frac{\partial^2\phi}{\partial x^2}=0,\ \varepsilon_y=\frac{\partial v}{\partial y}=\frac{\partial^2\phi}{\partial y^2}=(1+\nu)\alpha T_0\left(1-\frac{y^2}{b^2}\right),\ \varepsilon_{xy}=0$$

$$\Rightarrow \boxed{\sigma'_x=\frac{2G}{1-\nu}\left[\cancel{\varepsilon_x}+\nu\varepsilon_y-(1+\nu)\alpha T\right]=-E\alpha T_0\left(1-\frac{y^2}{b^2}\right),\sigma'_y=\frac{2G}{1-\nu}\left[\varepsilon_y+\nu\cancel{\varepsilon_x}-(1+\nu)\alpha T\right]=0,\tau'_{xy}=0}$$

- The resultant surface forces



- To satisfy the zero tractions BCs, impose the load



- The exact solution to the homogeneous problem (above right) is very difficult.

Thermal Stresses in an Elastic Thin-Plate

- For $a >> b$, we may again ask Saint-Venant for help.
- Replace the parabolic surface traction with an equivalent (uniformly distributed) surface load.
- As a result, **the homogeneous Airy Stress Function**

$$\psi = cy^2 \;\Rightarrow\; \sigma_x'' = \frac{\partial^2\psi}{\partial y^2} = 2c,\quad \sigma_y'' = \frac{\partial^2\psi}{\partial x^2} = 0,\quad \tau_{xy}'' = -\frac{\partial^2\psi}{\partial x\partial y} = 0$$

- Total stresses become

$$\sigma_x = \sigma_x' + \sigma_x'' = 2c - E\alpha T_0\left(1 - \frac{y^2}{b^2}\right),\quad \sigma_y = \sigma_y' + \sigma_y'' = 0,\quad \tau_{xy} = \tau_{xy}' + \tau_{xy}'' = 0$$

- Zero tractions require

$$(\sigma_x)_{x=\pm a} = 0,\quad (\tau_{xy})_{x=\pm a} = 0;\qquad (\sigma_y)_{y=\pm b} = 0,\quad (\tau_{xy})_{y=\pm b} = 0$$

- The last three are automatically satisfied.

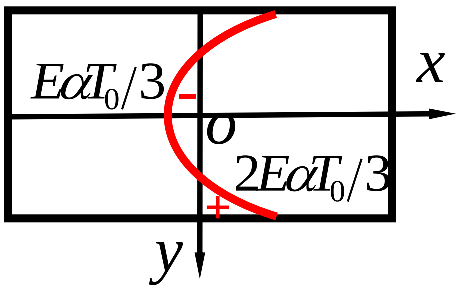
Thermal Stresses in an Elastic Thin-Plate

- The first condition cannot be satisfied in a pointwise sense.
- However, the resultant force of the surface traction must vanish (Saint-Venant's Principle)

$$\int_{-b}^{b}(\sigma_x)_{x=\pm a}\mathrm{d}y=0,\quad \int_{-b}^{b}(\sigma_x)_{x=\pm a}y\mathrm{d}y=0 \quad\Rightarrow\quad 2c=\frac{2}{3}E\alpha T_0$$

- The approximated thermal stresses

$$\sigma_x=E\alpha T_0\left(\frac{y^2}{b^2}-\frac{1}{3}\right),\quad \sigma_y=0,\quad \tau_{xy}=0$$



- Accurate for regions $2b$ away from the vertical boundaries.

Solution Strategy to Plane Thermoelasticity

- Determine the temperature variation from heat conduction and energy equation, if not given.
- Under either stress or displacement formulation, identify a particular solution due to temperature effects to the governing equation (the Beltrami-Michell or Navier).
- Evaluate the resultant stresses at the domain boundaries.
- Solve the corresponding isothermal problem, whose BCs are set as the general traction BCs subtracted by the contributions due to temperature effects.

Polar Coordinate

- Strain-Displacement relationship

$$\varepsilon_r=\frac{\partial u_r}{\partial r},\quad \varepsilon_\theta=\frac{1}{r}\left(u_r+\frac{\partial u_\theta}{\partial\theta}\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right).$$

- Hooke's law

$$\varepsilon_{\alpha\beta}=\frac{1}{2G}\left(\sigma_{\alpha\beta}-\frac{3-\kappa}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)+(1+\eta)\alpha T\delta_{\alpha\beta},\quad \sigma_{\alpha\beta}=2G\left[\varepsilon_{\alpha\beta}-\frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}+\frac{2(1+\eta)}{1-\kappa}\alpha T\delta_{\alpha\beta}\right]$$

$$\varepsilon_r=\frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_r-\frac{3-\kappa}{1+\kappa}\sigma_\theta\right)+(1+\eta)\alpha T,\varepsilon_\theta=\frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_\theta-\frac{3-\kappa}{1+\kappa}\sigma_r\right)+(1+\eta)\alpha T,\varepsilon_{r\theta}=\frac{1}{2G}\sigma_{r\theta}.$$

$$\sigma_r=-\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_r+(3-\kappa)\varepsilon_\theta-4(1+\eta)\alpha T\right],\sigma_\theta=-\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_\theta+(3-\kappa)\varepsilon_r-4(1+\eta)\alpha T\right],\tau_{r\theta}=2G\varepsilon_{r\theta}.$$

For plane strain: $\kappa=3-4\nu$ or $\nu=\frac{3-\kappa}{4}$, $\eta=\nu$; For plane stress: $\kappa=\frac{3-\nu}{1+\nu}$ or $\nu=\frac{3-\kappa}{1+\kappa}$, $\eta=0$.

- Equilibrium equations

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=0,\quad \frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{r\theta}}{r}=0.$$

Polar Coordinate – Stress Function Formulation

- Airy Stress Function Representation

$$\sigma_r = \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \psi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)$$

- Beltrami-Michell Equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi + \frac{8G(1+\eta)}{1+\kappa}\alpha\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)T = 0$$

For plane strain: $\frac{8G(1+\eta)}{1+\kappa} = \frac{E}{1-\nu}$	For plane stress: $\frac{8G(1+\eta)}{1+\kappa} = E$

- Axi-symmetric solution

$$\sigma_r = \frac{1}{r}\frac{d\psi}{dr}, \quad \sigma_\theta = \frac{d^2\psi}{dr^2} = \frac{d}{dr}\frac{d\psi}{dr} = \frac{d}{dr}(r\sigma_r), \quad \sigma_{r\theta} = 0$$

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right]\right\} + \frac{8G(1+\eta)\alpha}{1+\kappa}\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right)\right]\right\} = -\frac{8G(1+\eta)\alpha}{1+\kappa}\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

Polar Coordinate – Displacement Formulation

- Navier's (governing) equations without body forces

$$\nabla^2\boldsymbol{u}-\frac{2}{1-\kappa}\nabla(\nabla\cdot\boldsymbol{u})+\frac{4(1+\eta)}{1-\kappa}\alpha\nabla T=0.$$

$$\Rightarrow\begin{cases}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r^2}\right)-\frac{2}{1-\kappa}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{4(1+\eta)}{1-\kappa}\alpha\frac{\partial T}{\partial r}=0,\\ \left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right)-\frac{2}{1-\kappa}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{4(1+\eta)}{1-\kappa}\alpha\frac{1}{r}\frac{\partial T}{\partial\theta}=0.\end{cases}$$

- If the displacement is derivable from a potential

$$\boldsymbol{u}=\nabla\phi\ \Rightarrow\ \boxed{u_r=\frac{\partial\phi}{\partial r},\ u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}}\ \Rightarrow\ \boxed{\varepsilon_r=\frac{\partial^2\phi}{\partial r^2},\quad \varepsilon_\theta=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2},\quad \varepsilon_{r\theta}=\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}.}$$

$$\boxed{\nabla^2\phi=\varepsilon_r+\varepsilon_\theta=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi=\frac{4(1+\eta)}{1+\kappa}\alpha T}$$

For plane strain: $\kappa=3-4\nu,\ \eta=\nu:\ \ \nabla^2\phi=\frac{1+\nu}{1-\nu}\alpha T$	For plane stress: $\kappa=\frac{3-\nu}{1+\nu},\ \eta=0:\ \ \nabla^2\phi=(1+\nu)\alpha T$

Polar Coordinate – Displacement Formulation

- Particular solution + homogeneous solution

$$\phi = \phi^{(h)} + \phi^{(p)} \Rightarrow \nabla^2 \phi^{(h)} = 0,\ \nabla^2 \phi^{(p)} = \frac{4(1+\eta)}{1+\kappa} \alpha T$$

- **Homogeneous solution**

$$\varepsilon_r = \frac{\partial^2 \phi^{(h)}}{\partial r^2},\quad \varepsilon_\theta = \frac{1}{r}\frac{\partial \phi^{(h)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi^{(h)}}{\partial \theta^2},\quad \varepsilon_{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi^{(h)}}{\partial \theta}\right) \Rightarrow \varepsilon_{\gamma\gamma} = \varepsilon_r + \varepsilon_\theta = \nabla^2 \phi^{(h)} = 0$$

$$\sigma_{\alpha\beta} = 2G\left[\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)} \cancel{\varepsilon_{\gamma\gamma}}\, \delta_{\alpha\beta}\right] = 2G\varepsilon_{\alpha\beta} \Rightarrow$$

$$\sigma_r = 2G\varepsilon_r = 2G\frac{\partial^2 \phi^{(h)}}{\partial r^2},$$

$$\sigma_\theta = 2G\varepsilon_\theta = 2G\left(\frac{1}{r}\frac{\partial \phi^{(h)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi^{(h)}}{\partial \theta^2}\right)$$

$$\sigma_{r\theta} = 2G\varepsilon_{r\theta} = 2G\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi^{(h)}}{\partial \theta}\right)$$

Polar Coordinate – Displacement Formulation

- Axi-symmetric solution: $T = T(r)$

$$u_r = u_r(r),\ u_\theta = 0, \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r)\right) = \frac{4(1+\eta)}{1+\kappa}\alpha\frac{\mathrm{d}T}{\mathrm{d}r},}$$

$$\Rightarrow \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r) = \frac{4(1+\eta)}{1+\kappa}\alpha T + A_1, \quad \Rightarrow \boxed{u_r = \frac{4(1+\eta)}{(1+\kappa)}\frac{\alpha}{r}\int Tr\mathrm{d}r + A_1 r + \frac{A_2}{r},}$$

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r} = \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}, \quad \varepsilon_\theta = \frac{u_r}{r} = \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}, \quad \varepsilon_{r\theta} = 0$$

$$\sigma_r = -\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_r + (3-\kappa)\varepsilon_\theta - 4(1+\eta)\alpha T\right],$$

$$\sigma_\theta = -\frac{G}{1-\kappa}\left[(1+\kappa)\varepsilon_\theta + (3-\kappa)\varepsilon_r - 4(1+\eta)\alpha T\right], \quad \tau_{r\theta} = 0;$$

For plane strain: $\kappa = 3-4\nu$ or $\nu = \frac{3-\kappa}{4},\ \eta = \nu;$

For plane stress: $\kappa = \frac{3-\nu}{1+\nu}$ or $\nu = \frac{3-\kappa}{1+\kappa},\ \eta = 0.$

Polar Coordinate – Displacement Formulation

- Axi-symmetric solution: $T = T(r)$

For plane strain:	For plane stress:
$u_r = \frac{(1+\nu)}{(1-\nu)}\frac{\alpha}{r}\int Tr\mathrm{d}r + A_1 r + \frac{A_2}{r};$	$u_r = \frac{(1+\nu)\alpha}{r}\int Tr\mathrm{d}r + A_1 r + \frac{A_2}{r};$
$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u_r}{r};$	$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u_r}{r};$
$\sigma_r = \frac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha T\right],$	$\sigma_r = \frac{2G}{1-\nu}\left[\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha T\right],$
$\sigma_\theta = \frac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_\theta + \nu\varepsilon_r - (1+\nu)\alpha T\right].$	$\sigma_\theta = \frac{2G}{1-\nu}\left[\varepsilon_\theta + \nu\varepsilon_r - (1+\nu)\alpha T\right].$

Polar Coordinate – Stress Function Formulation

- Axi-symmetric solution

$$\Rightarrow \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma_r\right)\right]\right\} = -\frac{8G(1+\eta)\alpha}{1+\kappa}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma_r\right)\right] = -\frac{8G(1+\eta)\alpha}{1+\kappa}\left(\frac{\mathrm{d}T}{\mathrm{d}r}+\frac{C_1}{r}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma_r\right) = -\frac{8G(1+\eta)\alpha}{1+\kappa}\left(Tr + C_1 r\ln r + 2C_2 r\right)$$

$$\Rightarrow \sigma_r = -\frac{8G(1+\eta)}{1+\kappa}\alpha\left[\frac{1}{r^2}\int Tr\,\mathrm{d}r \cancel{+\frac{C_1}{4}\left(2\ln r - 1\right)} + C_2 + \frac{C_3}{r^2}\right]$$

- For domains that include the origin, C_1 and C_3 must vanish.
- The hoop stress is then derived from

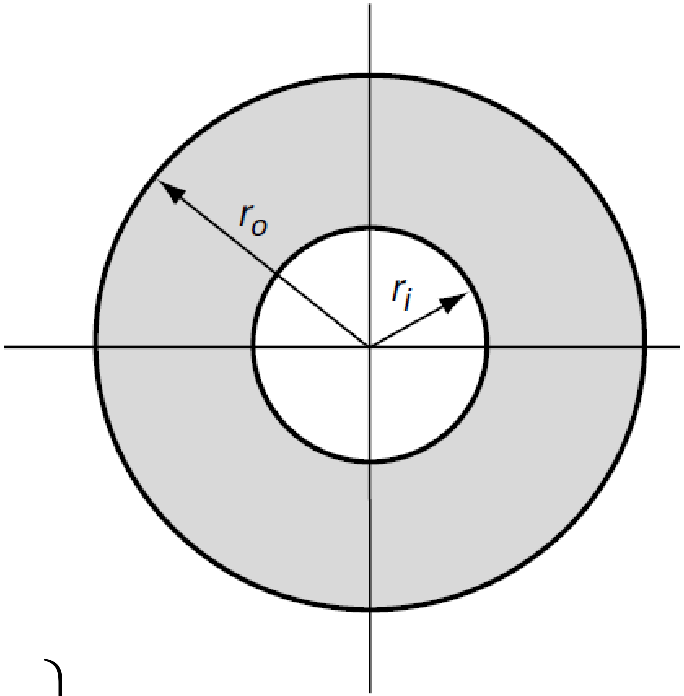
$$\sigma_\theta = \frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma_r\right)$$

- Since the stresses developed from the displacement solution do not contain the logarithmic term, it is inconsistent with single-valued displacements. **Drop it.**

Thermal Stresses in an Annular Circular Plate

- After dropping the logarithmic term, the stress formulation gives

$$\sigma_r = \frac{C_3}{r^2} + C_2 - \frac{E\alpha}{r^2}\int Tr\,\mathrm{d}r, \quad \sigma_\theta = \frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r)$$



- Zero tractions on boundaries

$$\sigma_r = \frac{E\alpha}{r^2}\left\{\frac{r^2 - r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} T(\xi)\xi\,\mathrm{d}\xi - \int_{r_i}^{r} T(\xi)\xi\,\mathrm{d}\xi\right\},$$

$$\sigma_\theta = \frac{E\alpha}{r^2}\left\{\frac{r^2 + r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} T(\xi)\xi\,\mathrm{d}\xi + \int_{r_i}^{r} T(\xi)\xi\,\mathrm{d}\xi - Tr^2\right\}.$$

- The displacement solution is

$$u_r = \frac{\alpha}{r}\left\{(1+\nu)\int_{r_i}^{r} T(\xi)\xi\,\mathrm{d}\xi + \frac{(1-\nu)r^2 + (1+\nu)r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} T(\xi)\xi\,\mathrm{d}\xi\right\}.$$

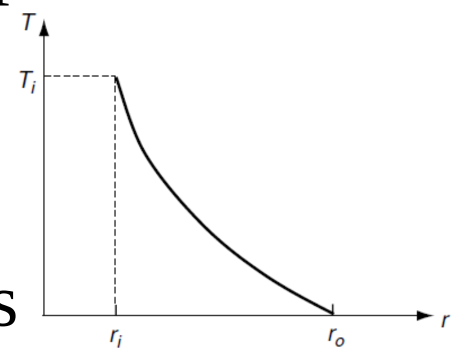
Thermal Stresses in an Annular Circular Plate

- To explicitly determine the stress distribution, the distribution of temperature variation must be determined.
- Assuming steady state conditions

$$\nabla^2 T = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \quad \Rightarrow \quad T = A_1 \ln r + A_2$$

- The two constants are determined from temperature BCs

$$T_{r=r_i} = T_i,\ T_{r=r_o} = 0 \ \Rightarrow \ T = \frac{T_i}{\ln(r_i/r_o)}\ln\frac{r}{r_o} = \frac{T_i}{\ln(r_o/r_i)}\ln\frac{r_o}{r}$$



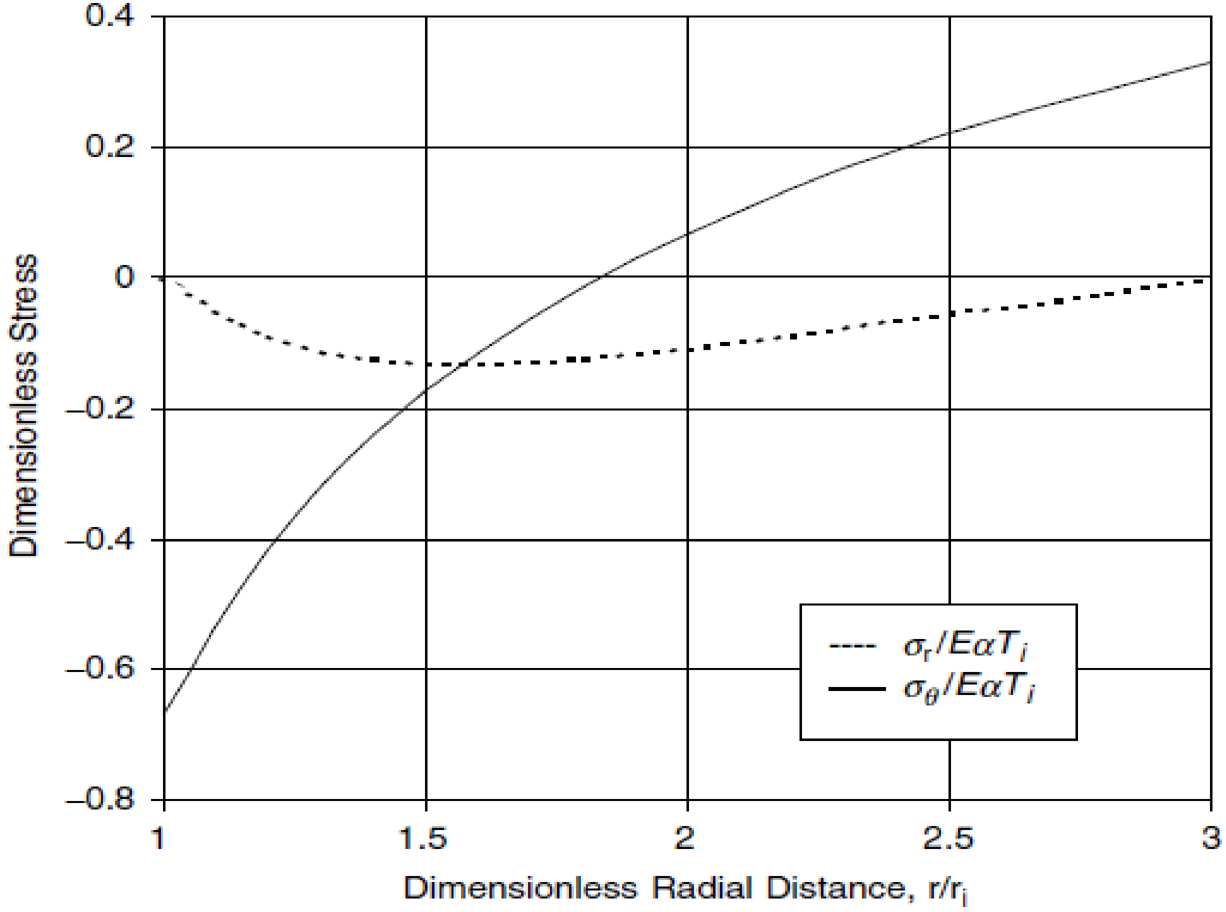
- Substituting the temperature back to stresses

$$\sigma_r = \frac{E\alpha T_i}{2\ln(r_o/r_i)}\left\{-\ln\frac{r_o}{r} - \frac{r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)\ln\frac{r_o}{r_i}\right\}$$

$$\sigma_\theta = \frac{E\alpha T_i}{2\ln(r_o/r_i)}\left\{1 - \ln\frac{r_o}{r} - \frac{r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)\ln\frac{r_o}{r_i}\right\}$$

Thermal Stresses in an Annular Circular Plate

- A numerical example with $r_o/r_i = 3$



Outline

- Heat Conduction Equation
- General 3-D Formulation
- Combined Plane Hooke's Law
- Stress Compatibility and Airy Stress Function
- Displacement Equilibrium and Displacement Potentials
- Thermal Stresses in Thin-Plates
- Summary of Solution Strategy
- Polar Coordinate: Airy Stress Function
- Polar Coordinate: Displacement Potentials
- Axi-symmetric Problems – Direct Solution
- Thermal Stresses in Circular Plates