Bending of Thin Plates
Outline

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• Relations between Internal Force and Stress
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Introduction

- One dimension (the thickness) is significantly smaller than the other two. 
  \((\frac{1}{8}-\frac{1}{5}) > \frac{t}{b} > (\frac{1}{80}-\frac{1}{100})\)
- Middle Surface: \(z = 0\).
- Only subjected to transvers loads.
- If a plate is only subjected to longitudinal loads, the problem is reduced to plane stress state.
- The bending problem of thin plates is analyzed with strategies similar to those of elastic beams.
Review of the Elementary Beam Theory

- Plane sections normal to the longitudinal axis of the beam remain planar.
- Only uniaxial longitudinal stress is assumed.

\[
E I \frac{d^2 w}{dx^2} = M, \quad \frac{d^2}{dx^2} \left( E I \frac{d^2 w}{dx^2} \right) = q
\]
Assumptions

- Straight lines normal to the middle surface remain straight and the same length.
- Stress components acting on planes parallel to the middle surface are significantly smaller than other components. The corresponding strain can therefore be neglected.

\[
0 = \varepsilon_z = \frac{\partial w}{\partial z} \quad \Rightarrow \quad w = w(x, y)
\]

\[
0 = \varepsilon_{zx} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad \Rightarrow \quad \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}
\]

\[
0 = \varepsilon_{zy} = \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \quad \Rightarrow \quad \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}
\]

Discard: \( \varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E}, \) \( \varepsilon_{zx} = \frac{1}{2G}\tau_{zx}, \) \( \varepsilon_{zy} = \frac{1}{2G}\tau_{zy}. \)
Assumptions

- Constitutive relations

\[
\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y), \quad \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x), \quad \varepsilon_{xy} = \frac{1}{2G} \tau_{xy}.
\]

- The middle surface of the plate is not strained during bending.

\[
\begin{align*}
\begin{cases}
(u)_{z=0} &= 0 \\
(v)_{z=0} &= 0
\end{cases} \quad \Rightarrow \quad \begin{cases}
(\varepsilon_x)_{z=0} &= \left(\frac{\partial u}{\partial x}\right)_{z=0} = 0 \\
(\varepsilon_y)_{z=0} &= \left(\frac{\partial v}{\partial y}\right)_{z=0} = 0 \\
(\varepsilon_{xy})_{z=0} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_{z=0} = 0
\end{cases}
\end{align*}
\]
Governing Equation in terms of Deflection $w(x, y)$

- Longitudinal displacements formulated in terms of the vertical deflection $w = w(x, y)$

\[
\begin{align*}
\frac{\partial u}{\partial z} &= -\frac{\partial w}{\partial x} \\
\frac{\partial v}{\partial z} &= -\frac{\partial w}{\partial y}
\end{align*}
\]

$\Rightarrow$

\[
\begin{align*}
u &= -\frac{\partial w}{\partial x} z + f_1(x, y) \\
v &= -\frac{\partial w}{\partial y} z + f_2(x, y)
\end{align*}
\]

\[
\begin{align*}
(u)_{z=0} &= 0 \\
(v)_{z=0} &= 0
\end{align*}
\]

$\Rightarrow$

\[
\begin{align*}
f_1(x, y) &= 0 \\
f_2(x, y) &= 0
\end{align*}
\]

- Longitudinal strains in terms of $w$

\[
\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\partial^2 w}{\partial x^2} z, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\partial^2 w}{\partial y^2} z, \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = -\frac{\partial^2 w}{\partial x \partial y} z
\]
Governing Equation in terms of Deflection $w(x, y)$

- Longitudinal stresses in terms of $w$

\[
\begin{aligned}
\sigma_x &= \frac{E}{1 - \nu^2} (\varepsilon_x + \nu \varepsilon_y) \\
\sigma_y &= \frac{E}{1 - \nu^2} (\varepsilon_y + \nu \varepsilon_x) \\
\tau_{xy} &= \frac{E}{(1 + \nu)} \varepsilon_{xy}
\end{aligned}
\]

\[
\begin{aligned}
\sigma_x &= -\frac{Ez}{1 - \nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\
\sigma_y &= -\frac{Ez}{1 - \nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
\tau_{xy} &= -\frac{Ez}{1 + \nu} \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\]

- Transvers shear stresses in terms of $w$

\[
\begin{aligned}
\frac{\partial \tau_{zx}}{\partial z} &= -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} \\
\frac{\partial \tau_{zy}}{\partial z} &= -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} \\
\frac{\partial \tau_{zx}}{\partial z} &= \frac{Ez}{1 - \nu^2} \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) = \frac{Ez}{1 - \nu^2} \frac{\partial}{\partial x} \nabla^2 w \\
\frac{\partial \tau_{zy}}{\partial z} &= \frac{Ez}{1 - \nu^2} \left( \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y \partial x^2} \right) = \frac{Ez}{1 - \nu^2} \frac{\partial}{\partial y} \nabla^2 w
\end{aligned}
\]

Integrate w.r.t $z$ ...
Governing Equation in terms of Deflection \( w(x, y) \)

- Transvers shear stresses in terms of \( w \)
  \[
  \tau_{zx} = \frac{Ez^2}{2(1-\nu^2)} \frac{\partial}{\partial x} \nabla^2 w + F_1(x, y), \quad \tau_{zy} = \frac{Ez^2}{2(1-\nu^2)} \frac{\partial}{\partial y} \nabla^2 w + F_2(x, y)
  \]

- Applying the BCs at the top/bottom surface
  \[
  \begin{align*}
  (\tau_{zx})_{z=\pm t/2} &= 0 \\
  (\tau_{zy})_{z=\pm t/2} &= 0
  \end{align*}
  \Rightarrow
  \begin{align*}
  \tau_{zx} &= \frac{E}{2(1-\nu^2)} \left(z^2 - \frac{t^2}{4}\right) \frac{\partial}{\partial x} \nabla^2 w \\
  \tau_{zy} &= \frac{E}{2(1-\nu^2)} \left(z^2 - \frac{t^2}{4}\right) \frac{\partial}{\partial y} \nabla^2 w
  \end{align*}
  \]

- Transvers normal stress in terms of \( w \)
  \[
  \frac{\partial \sigma_z}{\partial z} = - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} = \frac{E}{2(1-\nu^2)} \left(\frac{t^2}{4} - z^2\right) \nabla^4 w
  \]
  \[
  \Rightarrow \sigma_z = \frac{E}{2(1-\nu^2)} \left(\frac{t^2}{4} - \frac{z^3}{3}\right) \nabla^4 w + F_3(x, y)
  \]
Governing Equation in terms of Deflection \( w(x, y) \)

- Applying the BCs at the **bottom surface**

\[
(\sigma_z)_{z=t/2} = 0
\]

\[
\Rightarrow \sigma_z = \frac{E}{2(1-\nu^2)} \left[ \frac{t^2}{4} \left( z - \frac{t}{2} \right) - \frac{1}{3} \left( z^3 - \frac{t^3}{8} \right) \right] \nabla^4 w
\]

\[
\Rightarrow \sigma_z = -\frac{E}{6(1-\nu^2)} \left( z - \frac{t}{2} \right)^2 \left( z + t \right) \nabla^4 w
\]

- Further applying the BCs at the **top surface**

\[
(\sigma_z)_{z=-t/2} = -q \quad \Rightarrow \quad \frac{Et^3}{12(1-\nu^2)} \nabla^4 w = q
\]

\[
D \nabla^4 w = q, \quad D = \frac{Et^3}{12(1-\nu^2)}
\]

- **D**: Flexural Rigidity
**Internal Forces per Unit Length**

- **Definition:** It is customary to integrate the stresses over the (constant) plate thickness.

- **Design requirements**

- **Dealing with the Boundary Conditions (Saint-Venant BCs)**

\[ M_x = \int_{-t/2}^{t/2} z\sigma_x \, dz \]

\[ = -\frac{E}{1-v^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \int_{-t/2}^{t/2} z^2 \, dz \]

\[ = -\frac{Et^3}{12(1-v^2)} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]
Internal Forces \textbf{per Unit Length}

\begin{align*}
M_{xy} &= \int_{-t/2}^{t/2} z \tau_{xy} \, dz \\
&= -\frac{E}{1 + \nu} \frac{\partial^2 w}{\partial x \partial y} \int_{-t/2}^{t/2} z^2 \, dz \\
&= -\frac{Et^3}{12(1+\nu)} \frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \\
M_y &= \int_{-t/2}^{t/2} z \sigma_y \, dz \\
&= -\frac{Et^3}{12(1-\nu^2)} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
&= -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
M_{yx} &= M_{xy}
\end{align*}
Internal Forces per Unit Length

\[ Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} \, dz \]

\[ = \frac{E}{2(1 - \nu^2)} \frac{\partial}{\partial x} \nabla^2 w \int_{-\frac{t}{2}}^{\frac{t}{2}} \left( z^2 - \frac{t^2}{4} \right) \, dz \]

\[ = -\frac{Et^3}{12(1 - \nu^2)} \frac{\partial}{\partial x} \nabla^2 w = -D \frac{\partial}{\partial x} \nabla^2 w \]

\[ Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} \, dz = -\frac{Et^3}{12(1 - \nu^2)} \frac{\partial}{\partial y} \nabla^2 w \]

\[ = -D \frac{\partial}{\partial y} \nabla^2 w \]
Relations between Internal Forces and Stresses

\[ \sigma_x = -\frac{E_z}{1 - \nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]

\[ M_x = -\frac{E_t^3}{12(1 - \nu^2)} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]

\[ \sigma_y = -\frac{E_z}{1 - \nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \]

\[ M_y = -\frac{E_t^3}{12(1 - \nu^2)} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \]

\[ \tau_{xy} = -\frac{E_z}{1 + \nu} \frac{\partial^2 w}{\partial x \partial y} \]

\[ M_{xy} = -\frac{E_t^3}{12(1 + \nu)} \frac{\partial^2 w}{\partial x \partial y} \]
Relations between Internal Forces and Stresses

\[
\tau_{zx} = \frac{E}{2(1 - \nu^2)} \left( z^2 - \frac{t^2}{4} \right) \frac{\partial}{\partial x} \nabla^2 w
\]
\[
Q_x = -\frac{Et^3}{12(1 - \nu^2)} \frac{\partial}{\partial x} \nabla^2 w
\]
\[
\tau_{zy} = \frac{E}{2(1 - \nu^2)} \left( z^2 - \frac{t^2}{4} \right) \frac{\partial}{\partial y} \nabla^2 w
\]
\[
Q_y = -\frac{Et^3}{12(1 - \nu^2)} \frac{\partial}{\partial y} \nabla^2 w
\]
\[
\sigma_z = -\frac{E}{6(1 - \nu^2)} \left( z - \frac{t}{2} \right)^2 \left( z + t \right) \nabla^4 w
\]
\[
\frac{Et^3}{12(1 - \nu^2)} \nabla^4 w = q
\]

\[
\tau_{zx} = \frac{6}{t^3} \left( \frac{t^2}{4} - z^2 \right) Q_x
\]
\[
\tau_{zy} = \frac{6}{t^3} \left( \frac{t^2}{4} - z^2 \right) Q_y
\]
\[
\sigma_z = -2q \left( \frac{z}{t} - \frac{1}{2} \right)^2 \left( \frac{z}{t} + 1 \right)
\]
Differential Element Equilibrium

0 = \sum F_z = \frac{\partial Q_x}{\partial x} dx dy + \frac{\partial Q_y}{\partial y} dy dx + q dx dy \Rightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0

0 = \sum M_x = \frac{\partial M_{xy}}{\partial x} dx dy + \frac{\partial M_y}{\partial y} dy dx - Q_y dx dy - \frac{\partial Q_y}{\partial y} dy \frac{1}{2} dx \Rightarrow \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y

0 = \sum M_y \Rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \Rightarrow \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0
Boundary Conditions

- **Built-in / clamped edge** along $OA$
  \[
  (w)_{x=0} = 0, \quad \left( \frac{\partial w}{\partial x} \right)_{x=0} = 0
  \]

- **Simply supported edge** along $OC$
  \[
  0 = (w)_{y=0}, \quad 0 = (M_y)_{y=0} = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)_{y=0} \Rightarrow \left( \frac{\partial^2 w}{\partial y^2} \right)_{y=0} = 0
  \]

- **Completely free edges**, i.e. $BC$
  \[
  (M_x)_{x=a} = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)_{x=a} = 0, \quad (M_{xy})_{x=a} = 0, \quad (Q_x)_{x=a} = 0
  \]

The boundary conditions for a free edge were expressed by Poisson in this form.
Kirchhoff proved that the two requirements of Poisson dealing with the twisting moment $M_{xy}$ and with the shearing force $Q_x$ must be replaced by one condition.

Transforming every twisting moment into a force couple (Saint-Venant’s principle)

\[ \vec{V}_x \, dy = Q_x \, dy + \left( M_{xy} + \frac{\partial M_{xy}}{\partial y} \, dy \right) - M_x \]

\[ \Rightarrow \vec{V}_x = Q_x + \frac{\partial M_{xy}}{\partial y} \]
Boundary Conditions – Free Edges

- We are left with two concentrated forces at the corners $B$ and $C$

$$R_B = (M_{xy})_B \uparrow, \quad R_C = (M_{xy})_C \downarrow$$

- At the common corner $B$ of the Edges $AB$ and $BC$

$$R_B = (M_{xy})_B \uparrow + (M_{yx})_B \uparrow = 2(M_{xy})_B \uparrow$$

$$= -2D(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)_B$$

- For all four corners:
Boundary Equation Method – Elliptic Plate

- The boundary equation
  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \]

- Proposed deflection function
  \[ w = A \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2 \]

- On the boundary
  \[ w = 0, \quad \frac{\partial w}{\partial x} = \frac{4Ax}{a^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0, \quad \frac{\partial w}{\partial y} = \frac{4Ay}{b^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0 \]

This solution can only address elliptic plates with fixed boundary.
Boundary Equation Method – Elliptic Plate

• By the governing equation

\[
D \left( \frac{24A}{a^4} + \frac{16A}{a^2b^2} + \frac{24A}{b^4} \right) = q
\]

This solution can only address elliptic plates under constant pressure.

• The deflection

\[
A = \frac{qa^4b^4}{8D \left( 3a^4 + 2a^2b^2 + 3b^4 \right)} \implies w = \frac{qa^4b^4}{8D \left( 3a^4 + 2a^2b^2 + 3b^4 \right)} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2
\]

The applicability of this method is thus very limited.
Boundary Equation Method – Square Plate

- Consider a simply supported square plate subjected to sinusoidal load distribution
  \[ q(x, y) = q_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) \]

- The proposed deflection function
  \[ w(x, y) = w_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) \]
  This form satisfies both the BCs and governing equations.

  This solution works only for simply supported square plate.

- As an exercise, finish the problem by examining BCs, \( w_0 \), bending moments, shear forces, effective shear (reaction) forces at the edges, and corner forces. Check the force balance.
Fourier Method – Rectangular Plate

• Calculate the deflection of a **simply supported** rectangular plate, which is subjected to a distributed lateral load $q(x, y)$.

• The governing equation

\[
D \nabla^4 w = q, \quad D = \frac{Et^3}{12(1 - \nu^2)}
\]

• BCs

\[
\begin{align*}
(w)_{x=0} &= (w)_{x=a} = (w)_{y=0} = (w)_{y=b} = 0 \\
(M_x)_{x=0} &= (M_x)_{x=a} = (M_y)_{y=0} = (M_y)_{y=b} = 0
\end{align*}
\]
Fourier Method – Rectangular Plate

- Double Fourier Series solution
  \[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

- This solution satisfies all the BCs.

- By the governing equation
  \[ \pi^4 D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = q \]

- To derive the coefficients, expand \( q \) in Fourier series
  \[ q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]
  \[ = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \int_{0}^{a} \int_{0}^{b} q \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]
Fourier Method – Rectangular Plate

• Matching the coefficients $A_{mn}$ and $C_{mn}$

\[
A_{mn} = \frac{4 \int_0^a \int_0^b q \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy}{\pi^4 abD \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}
\]

• For constant pressure $q = q_0$

\[
A_{mn} = \frac{16q_0}{\pi^6 Dmn \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}, \quad m = 1, 3, 5, \ldots; \quad n = 1, 3, 5, \ldots
\]

\[
\Rightarrow w = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \frac{mn \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}{a^2 + b^2}
\]
Fourier Method – Rectangular Plate

• For concentrated load $F$ applied at $(\xi, \eta)$

\[ q(x, y) = \delta(x, y) = \begin{cases} \infty, & (x, y) = (\xi, \eta) \\ 0, & (x, y) \neq (\xi, \eta) \end{cases} \]

\[ \iint \delta(x, y) \, dx \, dy = F \]

\[ \iint \delta(x, y) \, f(x, y) \, dx \, dy = Ff(\xi, \eta) \]

where $f(x,y)$ should be sufficiently smooth.

• Fourier Series coefficients in $w$

\[ A_{mn} = \frac{4}{\pi^4 abD} \frac{4F}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi \xi}{a} \sin \frac{n\pi \eta}{b} \]

where $f(x,y)$ should be sufficiently smooth.

Two examples of Dirac delta functions
Summary

• The whole problem is formulated in terms of deflection $w$.

• The governing equation

$$D \nabla^4 w = q, \quad D = \frac{Et^3}{12(1-\nu^2)}$$

• Boundary conditions: three classical cases

• Built-in / clamped boundary: $w = 0, \quad \frac{\partial w}{\partial x} = 0$.

• Simply supported boundary: $w = 0, \quad M = 0$.

• Free edges: $M = 0, \quad Q + \frac{\partial M}{\partial t}^{nt} = 0$. 
Summary

- **Longitudinal displacements**

\[ u = -\frac{\partial w}{\partial x} z, \quad v = -\frac{\partial w}{\partial y} z \]

- **Stress field**

\[
\begin{align*}
\sigma_x &= -\frac{E_z}{1-v^2} \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \\
\sigma_y &= -\frac{E_z}{1-v^2} \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right) \\
\tau_{xy} &= -\frac{E_z}{1+v} \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]

\[
\begin{align*}
\tau_{zx} &= \frac{E}{2(1-v^2)} \left( z^2 - \frac{t^2}{4} \right) \frac{\partial}{\partial x} \nabla^2 w \\
\tau_{zy} &= \frac{E}{2(1-v^2)} \left( z^2 - \frac{t^2}{4} \right) \frac{\partial}{\partial y} \nabla^2 w \\
\sigma_z &= -\frac{E}{6(1-v^2)} \left( z - \frac{t}{2} \right)^2 (z + t) \nabla^4 w
\end{align*}
\]

- **Internal forces**

\[
\begin{align*}
M_x &= -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right), \\
M_y &= -D \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right), \\
M_{yx} &= M_{xy} = -D(1-v) \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]

\[
\begin{align*}
Q_x &= -D \frac{\partial}{\partial x} \nabla^2 w, \\
Q_y &= -D \frac{\partial}{\partial y} \nabla^2 w
\end{align*}
\]
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