Metal Plasticity

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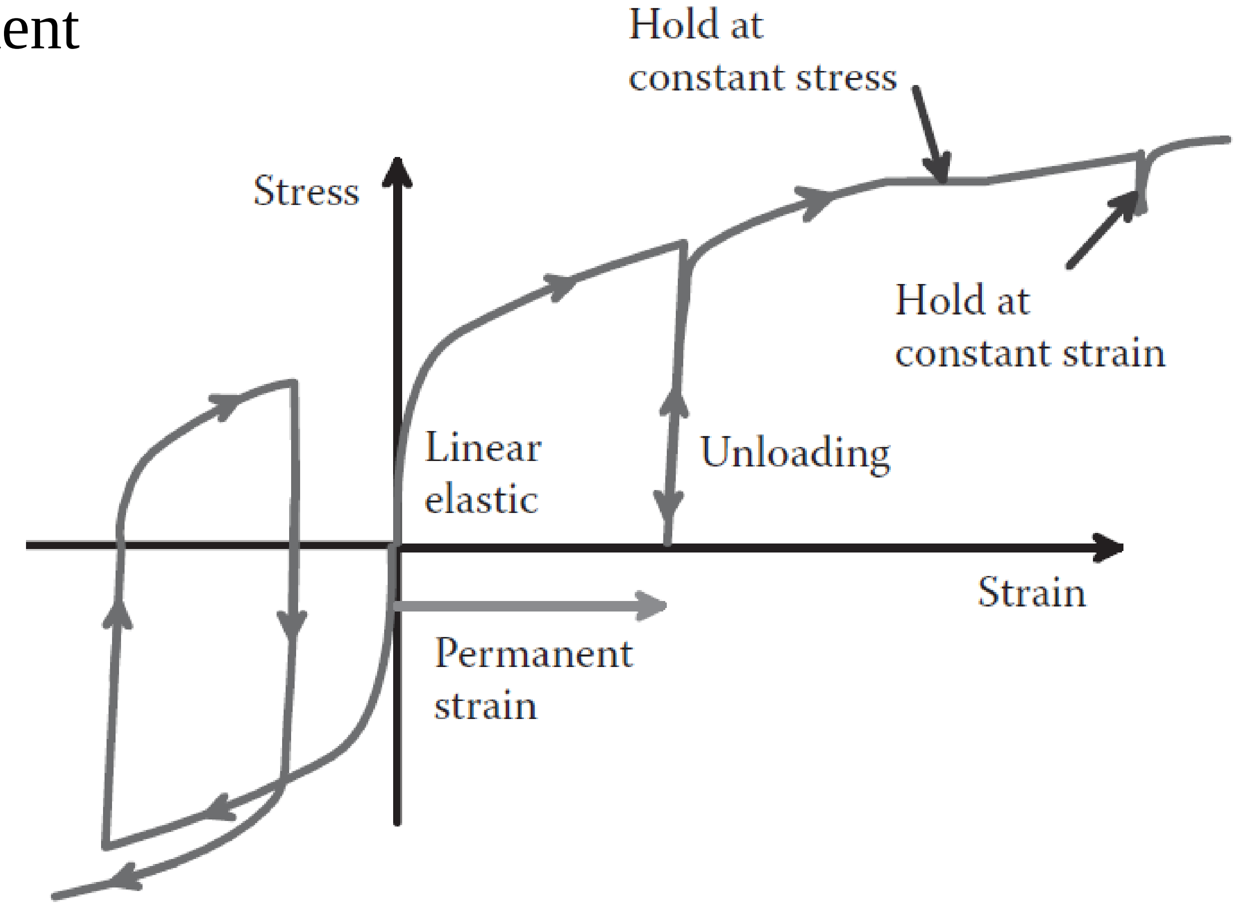
- Rough values of yield stress（常用工程材料屈服应力值）
- 3D plasticity theory（三维塑性理论）
- Decomposition of strain（应变分解）
- Yield criterion（屈服判据）
- Yield surface（屈服面）
- Isotropic strain hardening（各向同性强化）
- Kinematic strain hardening（运动强化）
- Principal of maximum plastic resistance（最大塑阻原理）
- Law of plastic flow（塑性流动定律）
- Elastic unloading conditions（弹性卸载的条件）

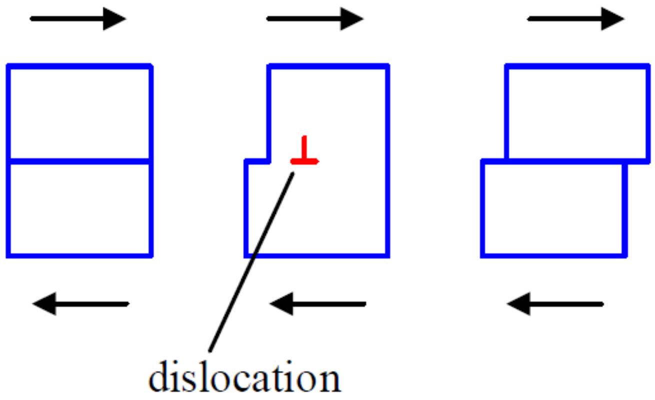
Introduction

- Linear elastic first; Plastic (permanent deformation); Unloading follows linear curve; Relaxation; Creep; Bauschinger effect; Cyclic hardening/softening; Rate, loading history and temperature dependent



Introduction

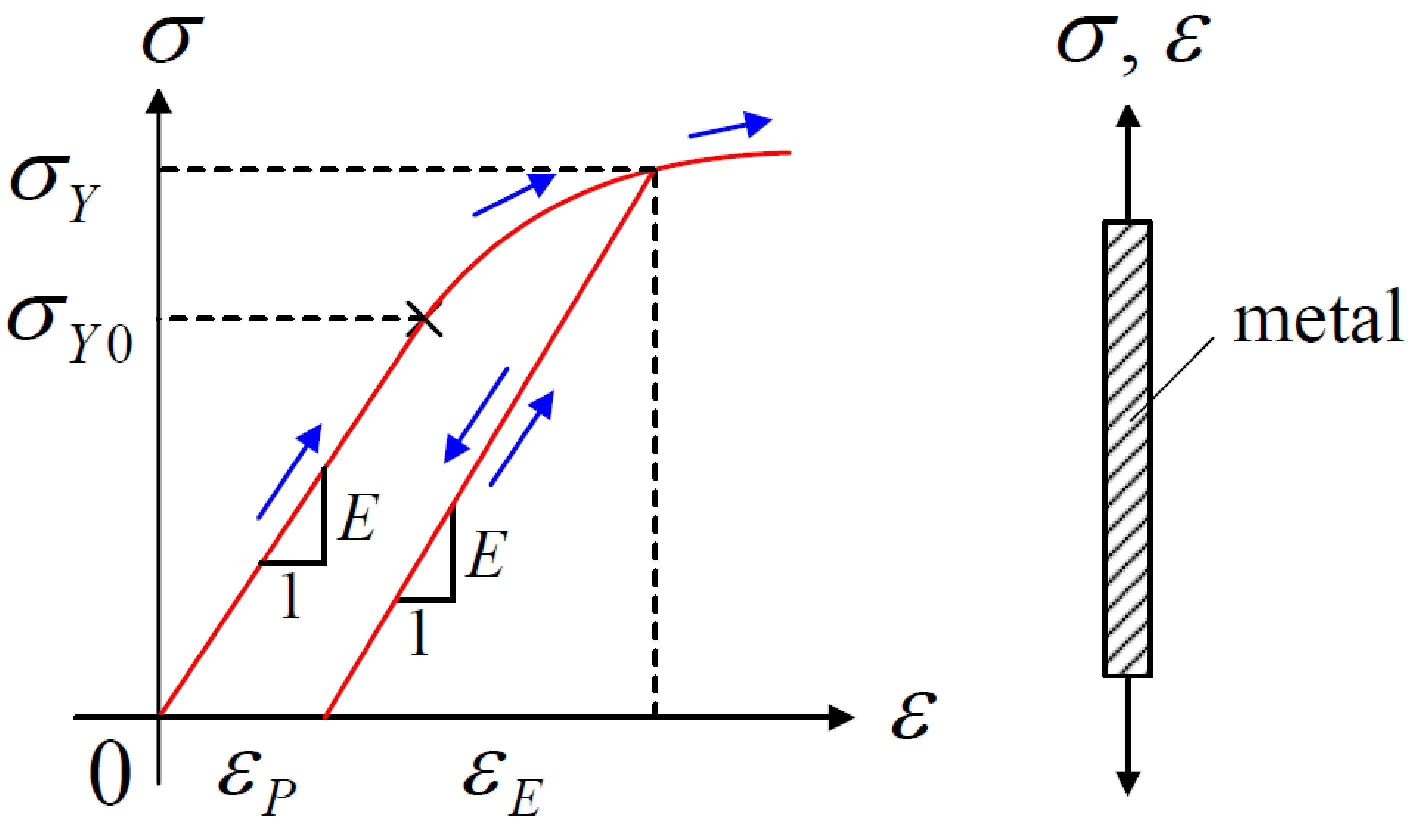
- In 1930's, Taylor and scientists experimentally measured the response of thin-walled tubes under combined torsion, axial loading, and hydrostatic pressure.
- Hydrostatic stress has no effects on plastic deformation.
- Plastic behavior doesn't induce volume change of a material.
- Plastic deformation is caused by shearing of atomic planes via propagation of a type of lattice defects called dislocations.
- During plastic loading, the principal components of the plastic strain rate tensor are parallel to the components of stress acting on the solid.
- **Levy–Mises flow rule** relates the principal plastic strain increment to the principal stresses.



$$\frac{d\varepsilon_I^p - d\varepsilon_{II}^p}{\sigma_I - \sigma_{II}} = \frac{d\varepsilon_{II}^p - d\varepsilon_{III}^p}{\sigma_{II} - \sigma_{III}} = \frac{d\varepsilon_{III}^p - d\varepsilon_I^p}{\sigma_{III} - \sigma_I}$$

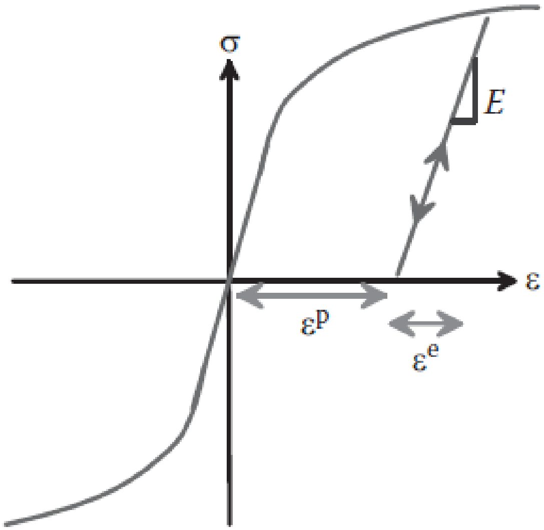
Introduction

- Decomposition of strain, yield criteria, strain hardening rules, plastic flow rule, elastic unloading criterion
- We restrict attention to small deformations (≤10%).

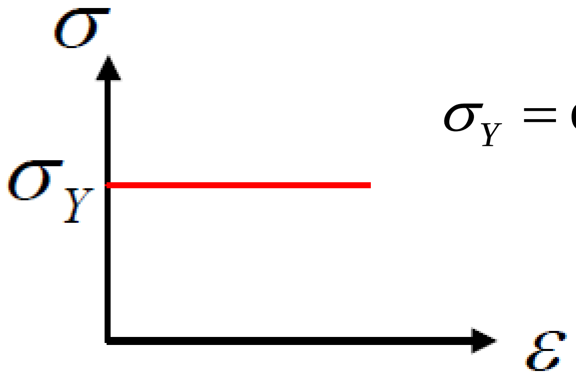


1D Plasticity

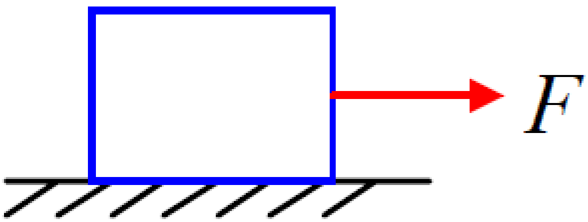
- **Decomposition of strain:** $d\varepsilon = d\varepsilon^e + d\varepsilon^p, \quad d\sigma = E d\varepsilon^e$
- **Yield criterion:** $\sigma = \sigma_Y\left[\varepsilon^p\right]$
- **Strain hardening rules** govern the functional dependence of yield stress on plastic strain.



✓Rigid-perfectly plastic model

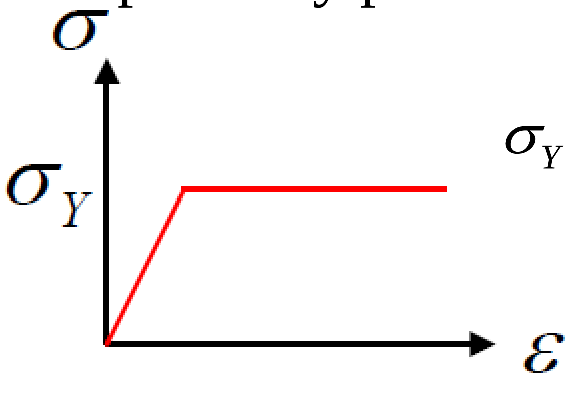


$\sigma_Y = \text{const.}$

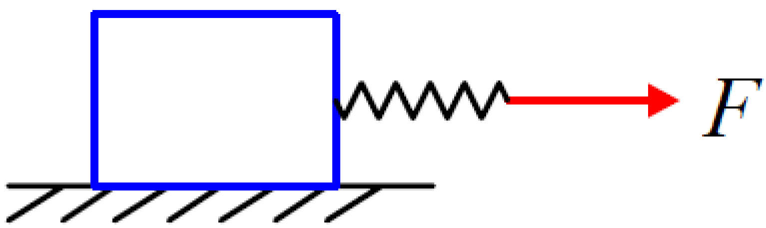


1D Plasticity

✓Elastic-perfectly plastic model

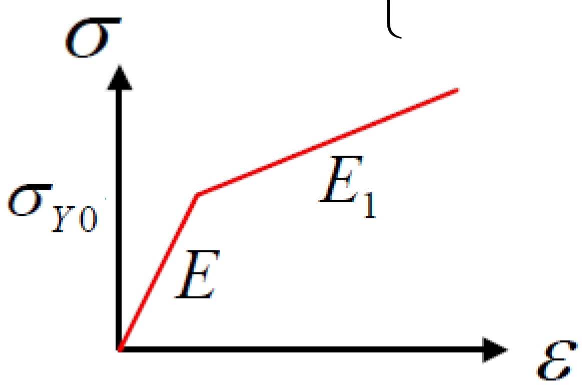


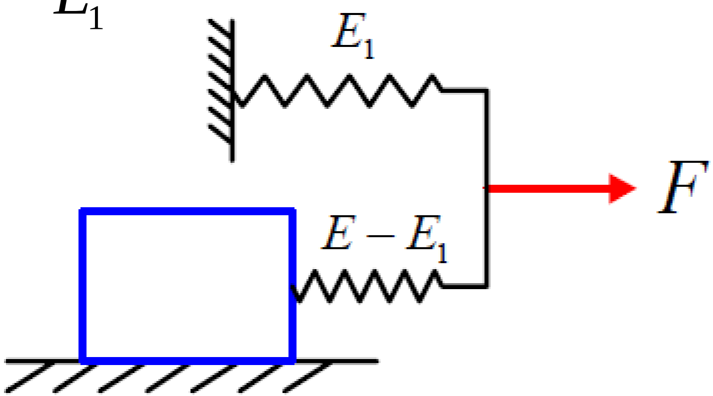
$$\sigma_Y = \text{const.}$$



✓Linear hardening model

$$\sigma_Y = \begin{cases} \sigma_{Y0}, & \sigma < \sigma_{Y0} \\ \sigma_Y = \sigma_{Y0} + \dfrac{EE_1}{E - E_1}\varepsilon^p, & \sigma \geq \sigma_{Y0} \end{cases}$$

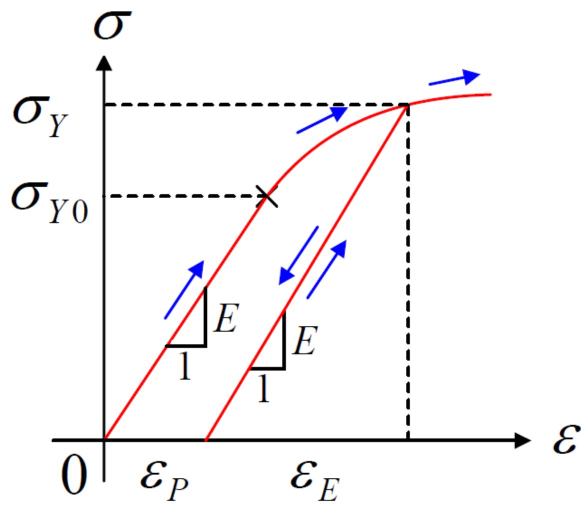




1D Plasticity

✓Power law hardening model

$$\sigma_Y = \begin{cases} \sigma_{Y0}, & \sigma < \sigma_{Y0} \\ \sigma_{Y0}\left(1+\dfrac{E\varepsilon^p}{\sigma_{Y0}}\right)^N, & \sigma \geq \sigma_{Y0} \end{cases}$$



- Here σ_{Y0}, E, and N can be treated as fitting parameters to experimental data. $0 \leq N < 1$ is called the hardening index.

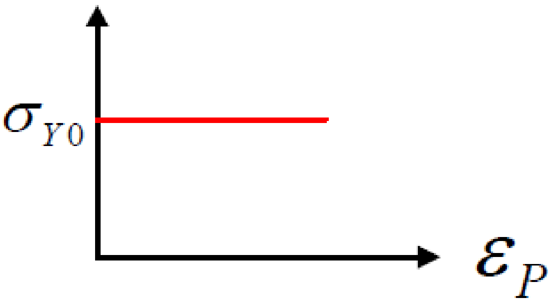
- Tangent modulus of the stress-plastic strain curve

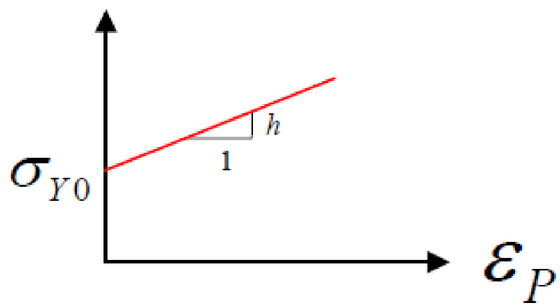
$$h = \frac{d\sigma}{d\varepsilon_P} = EN\left(1+\frac{E\varepsilon_P}{\sigma_{Y0}}\right)^{N-1}$$

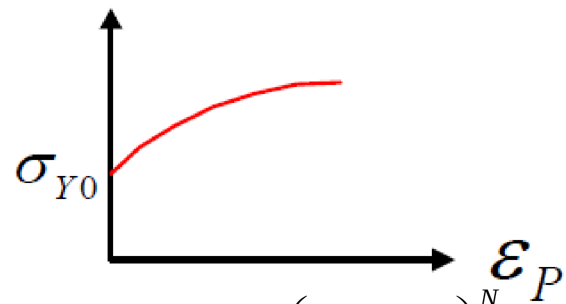
1D Plasticity

- **Law of plastic flow:**

$$d\varepsilon = d\varepsilon^{e} + d\varepsilon^{p} = \frac{d\sigma}{E} + \frac{d\sigma}{h}, \qquad \sigma = \sigma_Y, d\sigma > 0$$







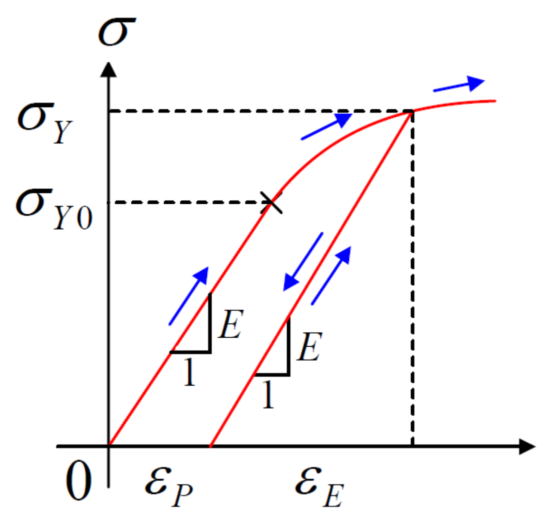
$h = 0$

$h = \text{const}$

$$\sigma_Y = \sigma_{Y0}\left(1 + \frac{E\varepsilon_P}{\sigma_{Y0}}\right)^{N}$$

- **Elastic unloading condition:**

$$\sigma d\sigma < 0$$



Rough Values of Yield Stress

Material	Yield Stress σ_Y / MNm^{-2}	Material	Yield Stress σ_Y / MNm^{-2}
Tungsten carbide	6000	Mild steel	220
Silicon carbide	10 000	Copper	60
Tungsten	2000	Titanium	180–1320
Alumina	5000	Silica glass	7200
Titanium carbide	4000	Aluminum and alloys	40–200
Silicon nitride	8000	Polyimides	52–90
Nickel	70	Nylon	49–87
Iron	50	PMMA	60–110
Low alloy steels	500–1980	Polycarbonate	55
Stainless steel	286–500	PVC	45–48

3D Plasticity Theory

- Decomposition of strain: $d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}, \quad d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}^{e}$
- **von Mises yield criterion:** take the distortional part of elastic strain energy as a criterion for the onset of plastic deformation

$$3J_2 = \frac{3}{2}\sigma'_{ij}\sigma'_{ij} = \frac{3}{2}\left(\sigma_I'^2 + \sigma_{II}'^2 + \sigma_{III}'^2\right) = \frac{3}{2}\left\{\sigma_I - \frac{1}{3}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right)\right\}^2 + \cdots + \cdots$$

$$= \frac{1}{2}\left\{\left(\sigma_I - \sigma_{II}\right)^2 + \left(\sigma_{II} - \sigma_{III}\right)^2 + \left(\sigma_{III} - \sigma_I\right)^2\right\}$$

$$1D\left(\sigma_I = \sigma_Y, \sigma_{II} = \sigma_{III} = 0\right): \quad \frac{3}{2}\sigma'_{ij}\sigma'_{ij} = \sigma_Y^2 \quad \Rightarrow \boxed{\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma_Y}$$

$$\boxed{\tau_8 = \frac{1}{3}\sqrt{\left(\sigma_I - \sigma_{II}\right)^2 + \left(\sigma_{II} - \sigma_{III}\right)^2 + \left(\sigma_{III} - \sigma_I\right)^2} = \frac{\sqrt{2}}{3}\sigma_e = \frac{1}{\sqrt{3}}\sigma'_{ij}\sigma'_{ij} = \frac{2}{\sqrt{3}}J_2}$$

- **von Mises yield function:**

$$f\left(\sigma_{ij}, \bar{\varepsilon}^p\right) = \sqrt{\frac{1}{2}\left\{\left(\sigma_I - \sigma_{II}\right)^2 + \left(\sigma_{II} - \sigma_{III}\right)^2 + \left(\sigma_{III} - \sigma_I\right)^2\right\}} - \sigma_Y\left[\bar{\varepsilon}^p\right] = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} - \sigma_Y\left[\bar{\varepsilon}^p\right] = 0.$$

Yield Criterion

- **Tresca yield criterion:**

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 : \quad \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_Y}{2} \quad \Rightarrow \sigma_1 - \sigma_3 = \sigma_Y$$

- **Tresca yield function:**

$$f\left(\sigma_{ij}, \bar{\varepsilon}^p\right) = \max\left\{\left|\sigma_I - \sigma_{II}\right|, \left|\sigma_{II} - \sigma_{III}\right|, \left|\sigma_I - \sigma_{III}\right|\right\} - \sigma_Y\left[\bar{\varepsilon}^p\right] = 0.$$

- Given the current stress σ_{ij} applied to the material, we need to determine current yield stress σ_Y based on an **effective plastic strain**:

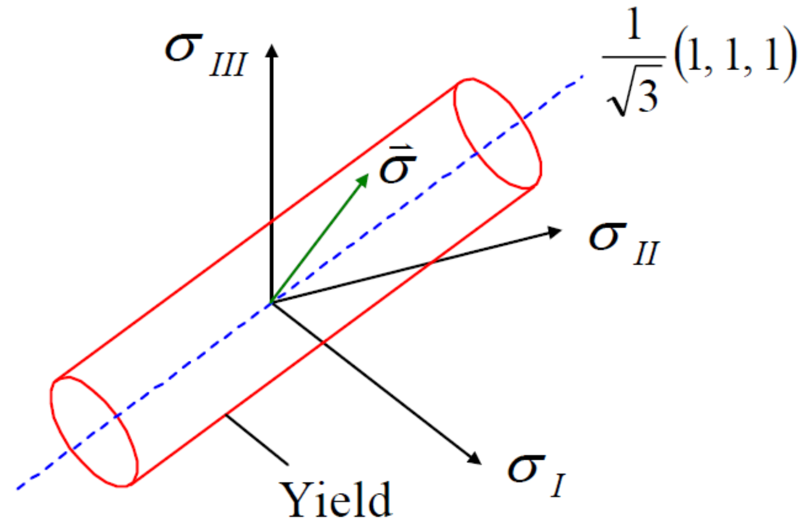
$$\boxed{d\bar{\varepsilon}^p = \sqrt{\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p}, \quad \bar{\varepsilon}^p = \int d\bar{\varepsilon}^p = \int \sqrt{\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p}}$$

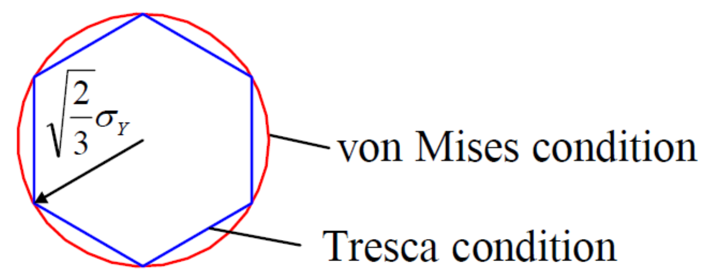
$$1D\left(\sigma_I = \sigma_Y, \sigma_{II} = \sigma_{III} = 0\right): d\bar{\varepsilon}^p = \sqrt{\frac{2}{3}\left(d\varepsilon_{11}^p d\varepsilon_{11}^p + d\varepsilon_{22}^p d\varepsilon_{22}^p + d\varepsilon_{33}^p d\varepsilon_{33}^p\right)}$$

$$= \sqrt{\frac{2}{3}\left(d\varepsilon_{11}^p d\varepsilon_{11}^p + 2\left(-\frac{1}{2} d\varepsilon_{11}^p\right)\left(-\frac{1}{2} d\varepsilon_{11}^p\right)\right)} = \sqrt{d\varepsilon_{11}^p d\varepsilon_{11}^p} = d\varepsilon_{11}^p$$

Yield Surface

- Geometric representation of von Mises yield condition in stress space.
- If the state of stress falls within the cylinder, the material is below yield and responds elastically.
- If the state of stress lies on the surface of the cylinder, the material yields and deforms plastically.
- The stress state cannot lie outside the cylinder; this would lead to an infinite plastic strain.

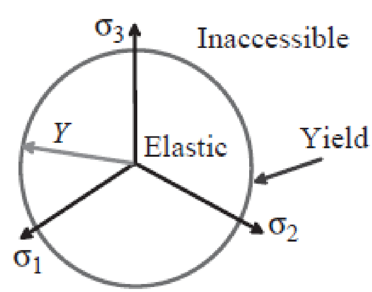


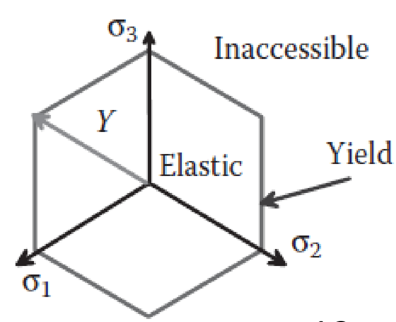


i.e.: $\sigma_I = \frac{R}{\sqrt{2}}, \sigma_{II} = -\frac{R}{\sqrt{2}}, \sigma_{III} = 0$

$$\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right\}}$$

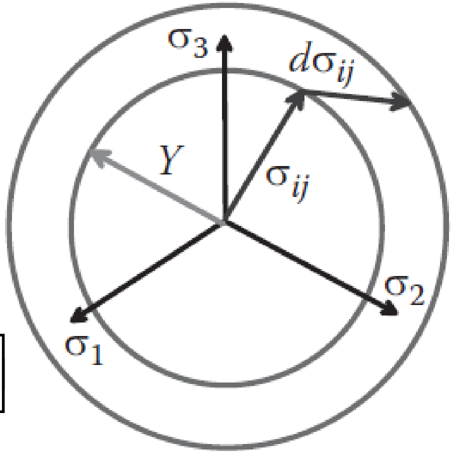
$$= R\sqrt{\frac{1}{2}\left\{2 + \frac{1}{2} + \frac{1}{2}\right\}} = \sqrt{\frac{3}{2}}R = \sigma_Y \Rightarrow R = \sqrt{\frac{2}{3}}\sigma_Y$$





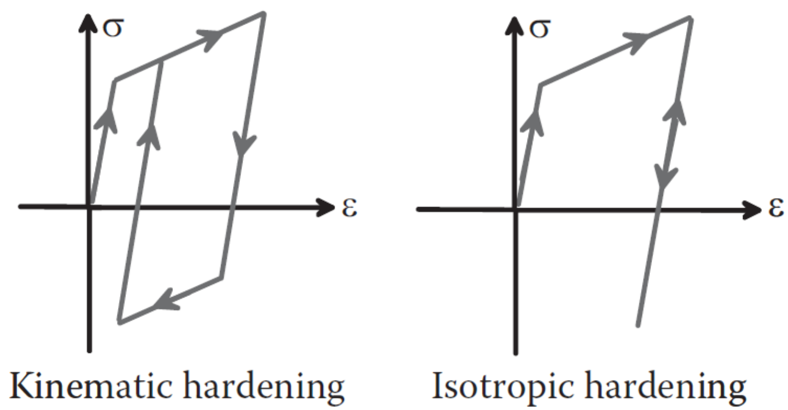
Isotropic Strain Hardening

- Plastic deformation typically causes the metal to strain harden, as obviously seen in 1D.
- Strain hardening can be modeled by relating the size and shape of the yield surface to plastic strain in some appropriate way.
- The easiest way to model strain hardening is to make the yield surface increase in size but remain the same shape.
- Determine the updated radius via the effective plastic strain, 1D hardening functions, and new yield stress.

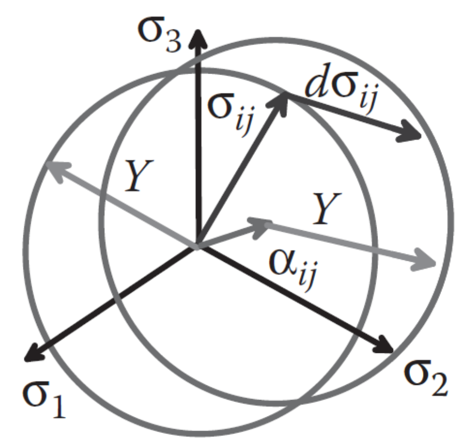


$$d\bar{\varepsilon}^{p} = \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}},\ \bar{\varepsilon}^{p} = \int d\bar{\varepsilon}^{p} = \int \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}},\ \sigma = \sigma_{Y}\left[\varepsilon^{p}\right]$$

Kinematic Strain Hardening



- An **isotropic hardening** law does not account for the Bauschinger effect.
- **Kinematic hardening** drag the yield surface in the direction of increasing stress as you deform the material in tension.



- This softens the material in compression, however.
- So, this law can model cyclic plastic deformation.

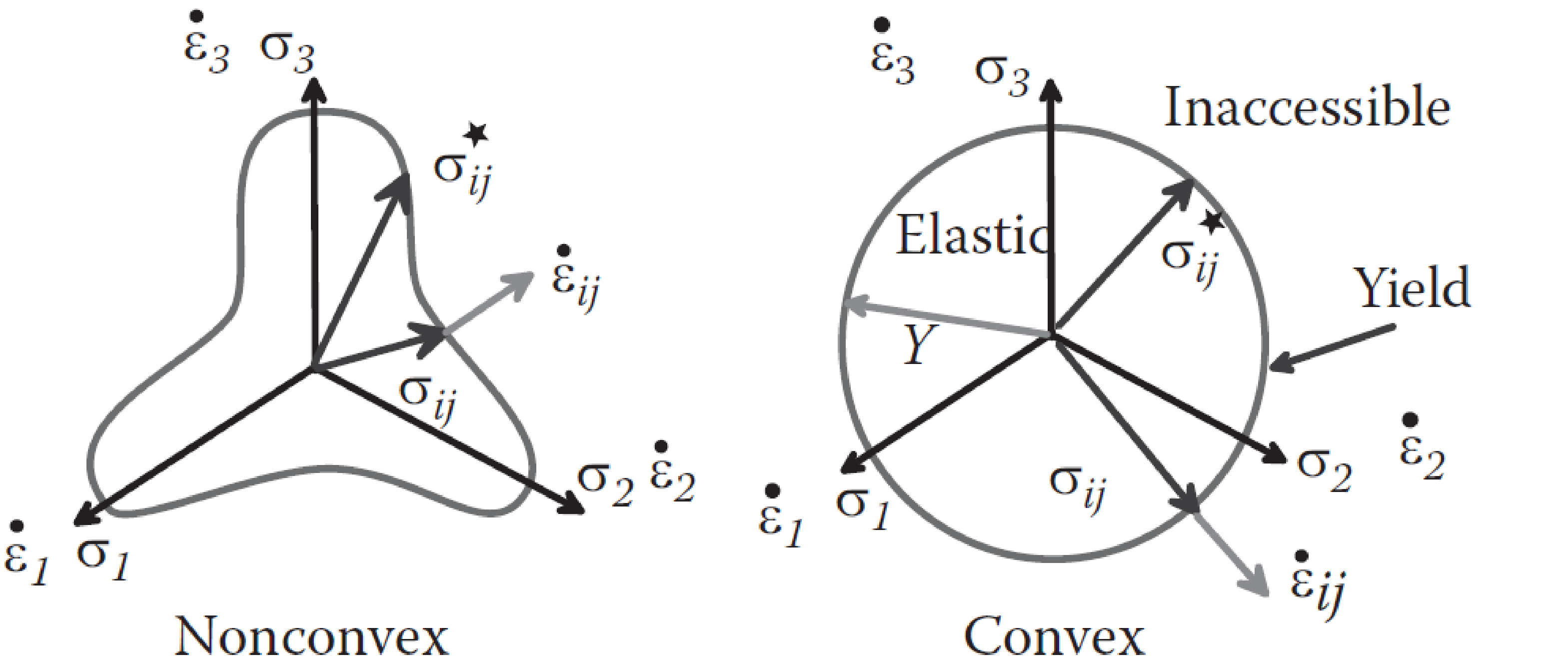
$$f\left(\sigma_{ij},\bar{\varepsilon}^{p}\right)=\sqrt{\frac{3}{2}\left(\sigma'_{ij}-\alpha_{ij}\left[\varepsilon^{p}_{ij}\right]\right)\left(\sigma'_{ij}-\alpha_{ij}\left[\varepsilon^{p}_{ij}\right]\right)}-\sigma_{Y0}=0.$$

- Linear kinematic hardening law: $d\alpha_{ij}=\frac{2}{3}c\,d\varepsilon^{p}_{ij}$

Principal of Maximum Plastic Resistance

$$\left(\sigma_{ij} - \sigma_{ij}^{*}\right) d\varepsilon_{ij}^{p} \geq 0$$

- The von Mises yield surface is convex: $d\varepsilon_{ij}^{p} \equiv d\bar{\varepsilon}^{p}\, \partial f / \partial \sigma_{ij}$
- The plastic strain increment is normal to the yield surface for plastically stable solids.



Law of Plastic Flow for Isotropic Hardening

- Experimental results (Levy-Mises theory) suggest that plastic strains can be derived from the yield criterion.
- Law of plastic flow for isotropic hardening:

$$\boxed{d\varepsilon_{ij}^{p} \equiv d\bar{\varepsilon}^{p}\frac{\partial f}{\partial \sigma_{ij}} = \left(d\bar{\varepsilon}^{p}\right)\left(\frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}\right) = \left(\frac{3}{2}\frac{1}{h}\frac{\sigma'_{kl}}{\sigma_Y}d\sigma_{kl}\right)\left(\frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}\right)}$$

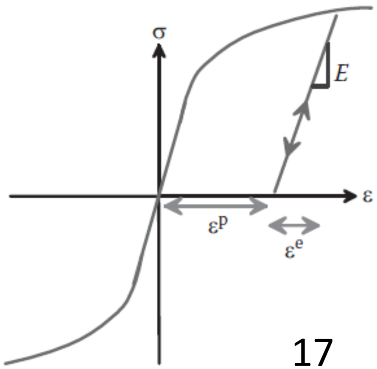
- Some intermediate results

$$f\left[\sigma_{ij},\bar{\varepsilon}^{p}\right] = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} - \sigma_Y\left[\bar{\varepsilon}^{p}\right] = 0 \quad \Rightarrow 0 = f\left[\sigma_{ij} + d\sigma_{ij}, \bar{\varepsilon}^{p} + d\bar{\varepsilon}^{p}\right] \approx \cancel{f\left[\sigma_{ij},\bar{\varepsilon}^{p}\right]} + \frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial f}{\partial \bar{\varepsilon}^{p}}d\bar{\varepsilon}^{p}$$

$$= \frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} - \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^{p}}d\bar{\varepsilon}^{p} = \frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} - h\,d\bar{\varepsilon}^{p} \quad \Rightarrow d\bar{\varepsilon}^{p} = \frac{1}{h}\frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} = \frac{1}{h}\frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}d\sigma_{ij}$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{2}\frac{1}{\sqrt{\frac{3}{2}\sigma'_{kl}\sigma'_{kl}}}\frac{3}{2}2\frac{\partial \sigma'_{pq}}{\partial \sigma_{ij}}\sigma'_{pq} = \frac{3}{2}\frac{1}{\sigma_Y}\left(\delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}\right)\sigma'_{pq} = \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}$$

$$\frac{\partial \sigma'_{pq}}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left(\sigma_{pq} - \frac{1}{3}\sigma_{kk}\delta_{pq}\right) = \delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ik}\delta_{jk}\delta_{pq} = \delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}$$



Law of Plastic Flow for Kinematic Hardening

- Law of plastic flow for kinematic hardening:

$$d\varepsilon_{ij}^{p} \equiv d\bar{\varepsilon}^{p}\frac{\partial f}{\partial \sigma_{ij}} = \left(d\bar{\varepsilon}^{p}\right)\left(\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right) = \left(\frac{3}{2c}\frac{\left(\sigma'_{kl}-\alpha_{kl}\right)}{\sigma_{Y0}}d\sigma_{kl}\right)\left(\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right)$$

- Some intermediate results

$$f\left[\sigma_{ij},\alpha_{ij}\right] = \sqrt{\frac{3}{2}\left(\sigma'_{ij}-\alpha_{ij}\right)\left(\sigma'_{ij}-\alpha_{ij}\right)} - \sigma_{Y0} = 0 \quad \Rightarrow 0 = f\left[\sigma_{ij}+d\sigma_{ij},\alpha_{ij}+d\alpha_{ij}\right] \approx \cancel{f\left[\sigma_{ij},\alpha_{ij}\right]} + \frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial f}{\partial \alpha_{ij}}d\alpha_{ij}$$

$$= \left(\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right)d\sigma_{ij} + \left(-\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right)\left(cd\bar{\varepsilon}^{p}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right) = \frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}d\sigma_{ij} - \underbrace{\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}^{2}}}_{=1}\left(cd\bar{\varepsilon}^{p}\right)$$

$$\Rightarrow \boxed{d\bar{\varepsilon}^{p} = \frac{3}{2c}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}d\sigma_{ij}}, \qquad d\alpha_{ij} = \frac{2}{3}cd\varepsilon_{ij}^{p} = \frac{2}{3}c\left(d\bar{\varepsilon}^{p}\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right) = cd\bar{\varepsilon}^{p}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}};$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \left(d\bar{\varepsilon}^{p}\right)\left(\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}\right); \qquad \frac{\partial f}{\partial \alpha_{ij}} = \frac{1}{2}\frac{1}{\sqrt{\frac{3}{2}\left(\sigma'_{kl}-\alpha_{kl}\right)\left(\sigma'_{kl}-\alpha_{kl}\right)}}\frac{3}{2}2\left(-\delta_{ip}\delta_{jq}\right)\left(\sigma'_{pq}-\alpha_{pq}\right) = -\frac{3}{2}\frac{\left(\sigma'_{ij}-\alpha_{ij}\right)}{\sigma_{Y0}}.$$

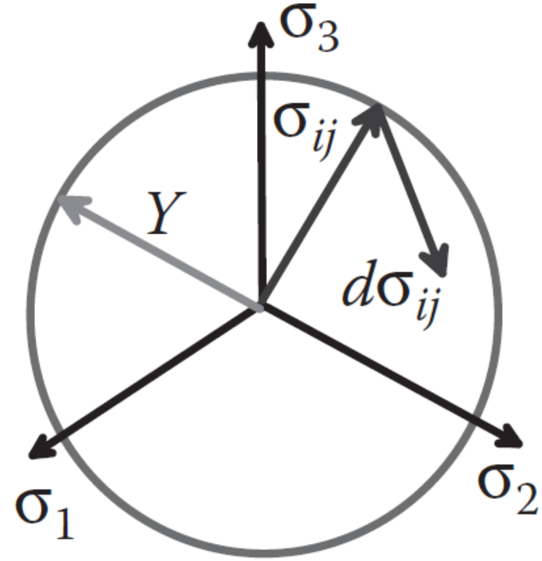
Elastic Unloading Conditions

- Both plastic flow laws are consistent with the Levy-Mises theory, which is based on experimental observations.

$$\frac{d\varepsilon_I^p - d\varepsilon_{II}^p}{\sigma_I - \sigma_{II}} = \frac{d\varepsilon_{II}^p - d\varepsilon_{III}^p}{\sigma_{II} - \sigma_{III}} = \frac{d\varepsilon_{III}^p - d\varepsilon_I^p}{\sigma_{III} - \sigma_I}$$

- **Elastic unloading conditions**

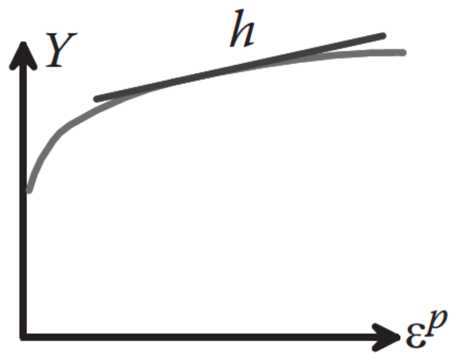
$$\sigma'_{ij} d\sigma_{ij} < 0, \qquad \left(\sigma'_{ij} - \alpha_{ij}\right) d\sigma_{ij} < 0.$$



- In both cases, the solid deforms elastically (no plastic strain) if the condition is satisfied.

Summary of 3D Plasticity Theory

- Summary of isotropically hardening elastic-plastic model
- Given: $E, \nu, \sigma_Y\left[\bar{\varepsilon}^p\right], h = d\sigma_Y / d\bar{\varepsilon}^p$



$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$$

$$d\varepsilon_{ij}^e = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij} = \frac{1}{2G} d\sigma'_{ij} + \frac{1}{9K}\delta_{ij} d\sigma_{kk}$$

$$d\varepsilon_{ij}^p = \begin{cases} 0, & \sigma_e - \sigma_Y\left[\bar{\varepsilon}^p\right] < 0 \\ \frac{3}{2}\frac{1}{h}\frac{\left\langle \sigma'_{kl} d\sigma_{kl} \right\rangle}{\sigma_Y}\frac{3}{2}\frac{\sigma'_{ij}}{\sigma_Y}, & \sigma_e - \sigma_Y\left[\bar{\varepsilon}^p\right] = 0 \end{cases}$$

where $\langle x \rangle = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

- It will correctly predict the conditions necessary to initiate yield under multiaxial loading.
- It will correctly predict the plastic strain rate under an arbitrary multiaxial stress state.
- It can model accurately any uniaxial stress-strain curve.

Summary of 3D Plasticity Theory

- Summary of linear kinematically hardening model
- Given: E, ν, σ_{Y0}, c

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

$$d\varepsilon_{ij}^{e} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij} = \frac{1}{2G} d\sigma'_{ij} + \frac{1}{9K}\delta_{ij} d\sigma_{kk}$$

$$d\varepsilon_{ij}^{p} = \begin{cases} 0, & \sqrt{\frac{3}{2}\left(\sigma'_{ij} - \alpha_{ij}\right)\left(\sigma'_{ij} - \alpha_{ij}\right)} - \sigma_{Y0} < 0 \\ \frac{3}{2c} \frac{\left\langle \left(\sigma'_{kl} - \alpha_{kl}\right) d\sigma_{kl} \right\rangle}{\sigma_{Y0}} \frac{3}{2} \frac{\left(\sigma'_{ij} - \alpha_{ij}\right)}{\sigma_{Y0}}, & \sqrt{\frac{3}{2}\left(\sigma'_{ij} - \alpha_{ij}\right)\left(\sigma'_{ij} - \alpha_{ij}\right)} - \sigma_{Y0} = 0 \end{cases}$$

- This constitutive equation is used primarily to model cyclic plastic deformation or plastic flow under nonproportional loading (in which principal axes of stress rotate significantly during plastic flow).