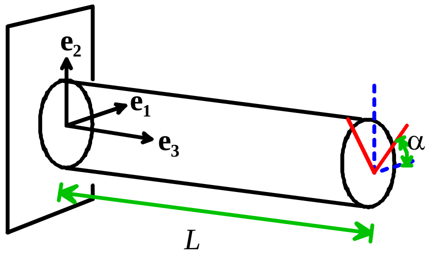
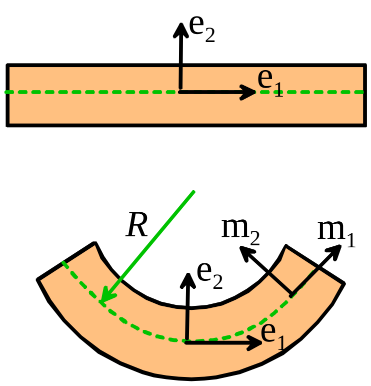
1. The displacement field in a homogeneous, isotropic circular shaft twisted through angle α at one end is given by



$$u_1 = x_1[\cos\left(\frac{\alpha x_3}{L}\right) - 1] - x_2 \sin\left(\frac{\alpha x_3}{L}\right)$$
$$u_2 = x_1 \sin\left(\frac{\alpha x_3}{L}\right) + x_2[\cos\left(\frac{\alpha x_3}{L}\right) - 1]$$
$$u_3 = 0$$

1) Calculate the matrix of components of the deformation gradient tensor
2) Calculate the matrix of components of the Lagrange strain tensor. Is the strain tensor a function of x_3? Why?
3) Find an expression for the increase in length of a material fiber of initial length *dl*, which is on the outer surface of the cylinder and initially oriented in the $\mathbf{e}_3$ direction.
4) Show that material fibers initially oriented in the $\mathbf{e}_2$ direction do not change their length.
5) Calculate the principal values and directions of the Lagrange strain tensor at the point $x_1 = a,\ \ x_2 = 0,\ \ x_3 = 0$. Hence, deduce the orientations of the material fibers that have the greatest and smallest increase in length.
6) Calculate the components of the infinitesimal strain tensor. Show that, for small values of α, the infinitesimal strain tensor is identical to the Lagrange strain tensor, but for finite rotations the two measures of deformation differ.
7) Use the infinitesimal strain tensor to obtain estimates for the lengths of material fibers initially oriented with the three basis vectors. Where is the error in this estimate greatest? How large can α be before the error in this estimate reaches 10%?

2. An initially straight beam is bent into a circle with radius *R* as shown in the figure. Material fibers that are perpendicular to the axis of the undeformed beam are assumed to remain perpendicular to the axis after deformation, and the beam's thickness and the length of its axis are assumed to be unchanged. Under these conditions the deformation can be described as



$$y_1 = (R - x_2)\sin(x_1 / R) \qquad y_2 = R - (R - x_2)\cos(x_1 / R)$$

where, as usual **x** is the position of a material particle in the undeformed beam, and **y** is the position of the same particle after deformation.

1) Calculate the deformation gradient field in the beam, expressing your answer as a function of x_1, x_2, and as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown.
2) Calculate the Lagrange strain field in the beam.
3) Calculate the infinitesimal strain field in the beam.

4) Compare the values of Lagrange strain and infinitesimal strain for two points that lie at $(x_1=0, x_2=h)$ and $(x_1=L, x_2=0)$. Explain briefly the physical origin of the difference between the two strain measures at each point. Recommend maximum allowable values of *h/R* and *L/R* for use of the infinitesimal strain measure in modeling beam deflections.

5) Calculate the deformed length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_1$ in the undeformed beam. Express your answer as a function of x_2.

6) Calculate the change in length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_2$ in the undeformed beam.

7) Show that the two material fibers described in 5） and 6) remain mutually perpendicular after deformation. Is this true for *all* material fibers that are mutually perpendicular in the undeformed solid?

8) Find the components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Left and Right stretch tensors $\mathbf{U}$ and $\mathbf{V}$ as well as the rotation tensor $\mathbf{R}$ for this deformation. You should be able to write down $\mathbf{U}$ and $\mathbf{R}$ by inspection, without needing to wade through the laborious general process outlined in class notes. The results can then be used to calculate $\mathbf{V}$.

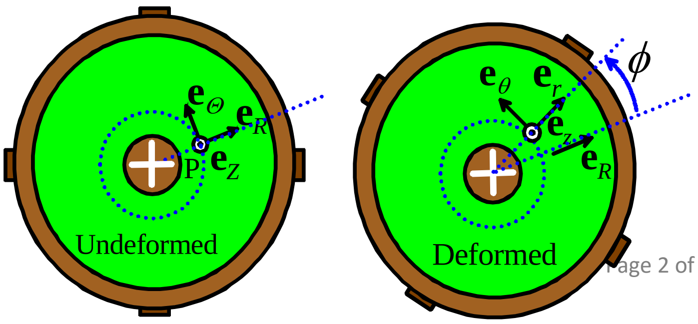
9) Find the principal directions of $\mathbf{U}$ as well as the principal stretches. You should be able to write these down using your physical intuition without doing any tedious calculations.

10) Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a basis in which $\mathbf{m}_1$ is parallel to the axis of the deformed beam, as shown in the figure. Write down the components of each of the unit vectors $\mathbf{m}_i$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, compute the transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ that is used to transform tensor components from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

11) Find the components of the deformation gradient tensor, Lagrange strain tensor, as well as $\mathbf{U}$ $\mathbf{V}$ and $\mathbf{R}$ in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

12) Find the principal directions of $\mathbf{V}$ expressed as components in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Again, you should be able to simply write down this result.

3. The figure shows a test designed to measure the response of a polymer to large shear strains. The sample is a hollow cylinder with internal radius a_0



and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated through an angle $\alpha(t)$. Assume that the specimen deforms as indicated in the figure, i.e. (a) cylindrical sections remain cylindrical; (b) no point in the specimen moves in the axial or radial directions; (c) that a cylindrical element of material at radius R rotates through angle $\phi(R,t)$ about the axis of the specimen. Take the undeformed configuration as reference. Let $(R,\boldsymbol{\Theta},Z)$ denote the cylindrical-polar coordinates of a material point in the reference configuration, and let $\{\mathbf{e}_R,\mathbf{e}_{\boldsymbol{\Theta}},\mathbf{e}_Z\}$ be cylindrical-polar basis vectors at $(R,\boldsymbol{\Theta},Z)$. Let (r,θ,z) denote the coordinates of this point in the deformed configuration, and let $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ by cylindrical-polar basis vectors located at (r,θ,z).

1) Write down expressions for (r,θ,z) in terms of $(R,\boldsymbol{\Theta},Z)$ (this constitutes the deformation mapping)
2) Let *P* denote the material point at $(R,\boldsymbol{\Theta},Z)$ in the reference configuration. Write down the reference position vector **X** of *P*, expressing your answer as components in the basis $\{\mathbf{e}_R,\mathbf{e}_{\boldsymbol{\Theta}},\mathbf{e}_Z\}$.
3) Write down the deformed position vector **x** of *P*, expressing your answer in terms of $(R,\boldsymbol{\Theta},Z)$ and basis vectors $\{\mathbf{e}_R,\mathbf{e}_{\boldsymbol{\Theta}},\mathbf{e}_Z\}$.
4) Find the components of the deformation gradient tensor **F** in $\{\mathbf{e}_R,\mathbf{e}_{\boldsymbol{\Theta}},\mathbf{e}_Z\}$. (Recall that the gradient operator in cylindrical-polar coordinates is $\nabla \equiv \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_{\boldsymbol{\Theta}} \frac{1}{R}\frac{\partial}{\partial \boldsymbol{\Theta}} + \mathbf{e}_Z \frac{\partial}{\partial Z} \right)$; recall also that $\frac{\partial \mathbf{e}_R}{\partial \Theta} = \mathbf{e}_\Theta$; $\frac{\partial \mathbf{e}_\Theta}{\partial \Theta} = -\mathbf{e}_R$)
5） Show that the deformation gradient can be decomposed into a sequence $\mathbf{F} = \mathbf{R} \cdot \mathbf{S}$ of a simple shear $\mathbf{S}$ followed by a rigid rotation through angle ϕ about the $\mathbf{e}_z$ direction **R.** In this case the simple shear deformation will have the form

$$\mathbf{S} = \mathbf{e}_R\mathbf{e}_R + \mathbf{e}_\Theta\mathbf{e}_\Theta + \mathbf{e}_Z\mathbf{e}_Z + k\mathbf{e}_\Theta\mathbf{e}_R$$

where k is to be determined.

6） Find the components of **F** in $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$.
7） Verify that the deformation is volume preserving (i.e. check the value of *J*=det(**F**))
8） Find the components of the right Cauchy-Green deformation tensors in $\{\mathbf{e}_R,\mathbf{e}_{\boldsymbol{\Theta}},\mathbf{e}_Z\}$

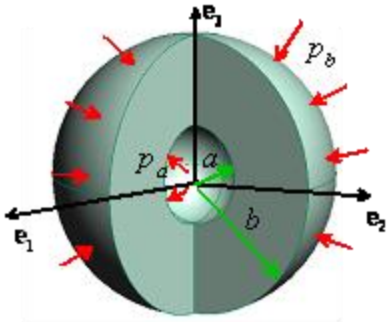
9） Find the components of the left Cauchy-Green deformation tensor in $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

10） Find $\mathbf{F}^{-1}$ in $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$.

11） Find the principal values of the stretch tensor **U**

12） Write down the velocity field **v** in terms of (r, θ, z) in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

13） Calculate the spatial velocity gradient **L** in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$



4. A spherical shell (see the figure) is made from an incompressible material. In its undeformed state, the inner and outer radii of the shell are A, B. After deformation, the new values are a, b. The deformation in the shell can be described (in Cartesian components) by the equation

$$y_i = \left(R^3 + a^3 - A^3\right)^{1/3} \frac{x_i}{R} \qquad R = \sqrt{x_k x_k}$$

1） Calculate the components of the deformation gradient tensor

2） Verify that the deformation is volume preserving

3） Find the deformed length of an infinitesimal radial line that has initial length l_0, expressed as a function of R

4） Find the deformed length of an infinitesimal circumferential line that has initial length l_0, expressed as a function of R

5） Using the results of 3) and 4), find the principal stretches for the deformation.

6） Find the *inverse* of the deformation gradient, expressed as a function of y_i. It is best to do this by working out a formula that enables you to calculate x_i in terms of y_i and $r = \sqrt{y_i y_i}$ and differentiate the result.

7） Recalculate 1) to 6), but this time solve the problem using spherical-polar coordinates, using the various formulas for vector and tensor operations given in lecture notes. In this case, you may assume that a point with position $\mathbf{x} = R\mathbf{e}_R$ in the undeformed solid has position vector

$$\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3} \mathbf{e}_R$$

after deformation.