Two-Dimensional Formulation

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- Plane Stress
- Boundary Conditions
- Correspondence between Plane Strain and Plane Stress
- Combined Plane Formulations
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- Airy Stress Function
- Polar Coordinate Formulation

Introduction

- Three-dimensional elasticity problems are very difficult to solve. Thus we will first solve a number of two-dimensional problems, and will explore three different theories:
  - *Plane Strain*
  - *Plane Stress*
  - *Anti-Plane Strain*

- Since all real elastic structures are three-dimensional, theories set forth here will be approximate models. The nature and accuracy of the approximation will depend on problem and loading geometry.

- The basic theories of *plane strain* and *plane stress* represent the fundamental plane problem in elasticity. While these two theories apply to significantly different types of two-dimensional bodies, their formulations yield very similar field equations.

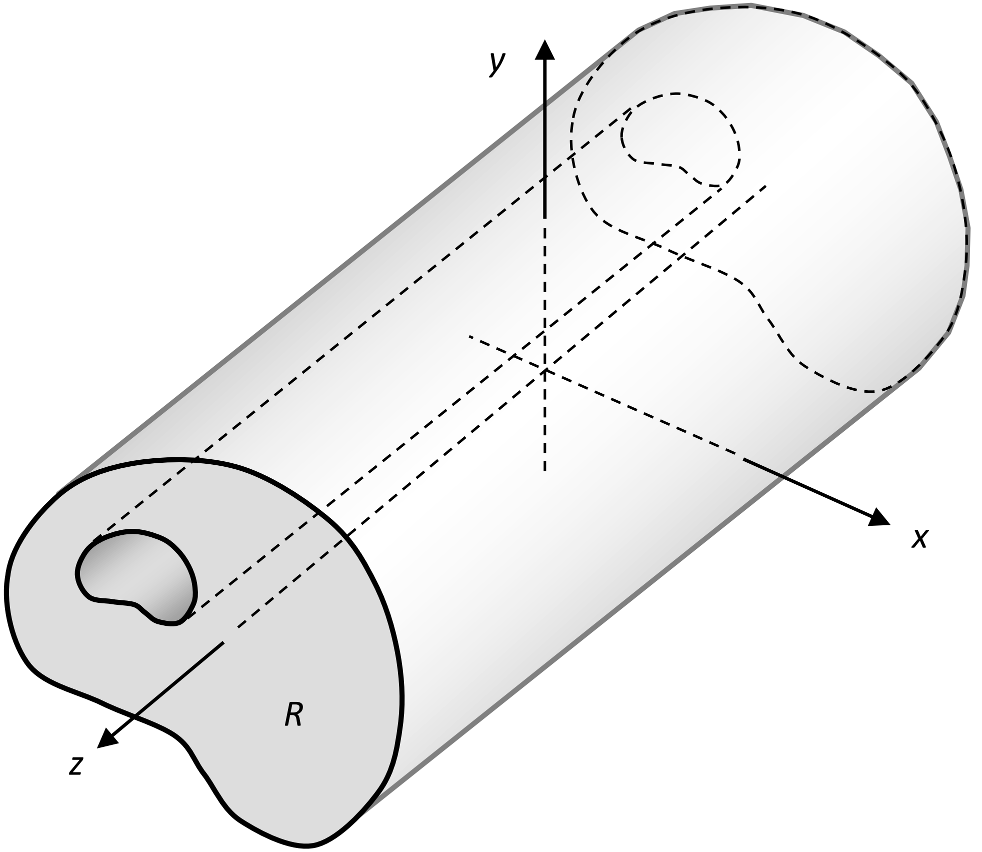
Plane Strain

- Consider an infinitely long cylindrical (prismatic) body as shown. If the body forces and tractions on lateral boundaries are independent of the z-coordinate and have no z-component, then the deformation field can be taken in the reduced form

$$u = u(x, y),$$

$$v = v(x, y),$$

$$w = 0.$$



Plane Strain Field Equations

- Displacement-strain relation: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \ \varepsilon_y = \frac{\partial v}{\partial y}, \ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_z = \frac{\partial w}{\partial z} = 0, \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0.$$

- Isotropic Hooke's Law: $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}; \ \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$

$$\sigma_x = \lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_x, \quad \sigma_y = \lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_y, \sigma_z = \lambda(\varepsilon_x + \varepsilon_y) = \nu(\sigma_x + \sigma_y)$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \quad \tau_{zx} = \tau_{zy} = 0$$

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = \frac{1+\nu}{E}(\sigma_x - \nu(\sigma_x + \sigma_y)),$$

$$\varepsilon_y = \frac{1+\nu}{E}\sigma_y - \frac{\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = \frac{1+\nu}{E}(\sigma_y - \nu(\sigma_x + \sigma_y)), \quad \varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Plane Strain Field Equations

- Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \cancel{\frac{\partial \tau_{xz}}{\partial z}} + F_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{yz}}{\partial z}} + F_y = 0,$$

$$\cancel{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}} + F_z = 0.$$

$$\Rightarrow$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0.$$

Plane Strain Field Equations

- Navier's Equations

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_x = 0,$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_y = 0,$$

$$G\nabla^2 w + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_z = 0.$$

$$\Rightarrow \quad G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0,$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0.$$

$$G\nabla^2 \boldsymbol{u} + (\lambda + G)\nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{F} = 0.$$

Plane Strain Field Equations

- Strain Compatibility $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$ (6 eqns)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

- Beltrami-Michell Equation:

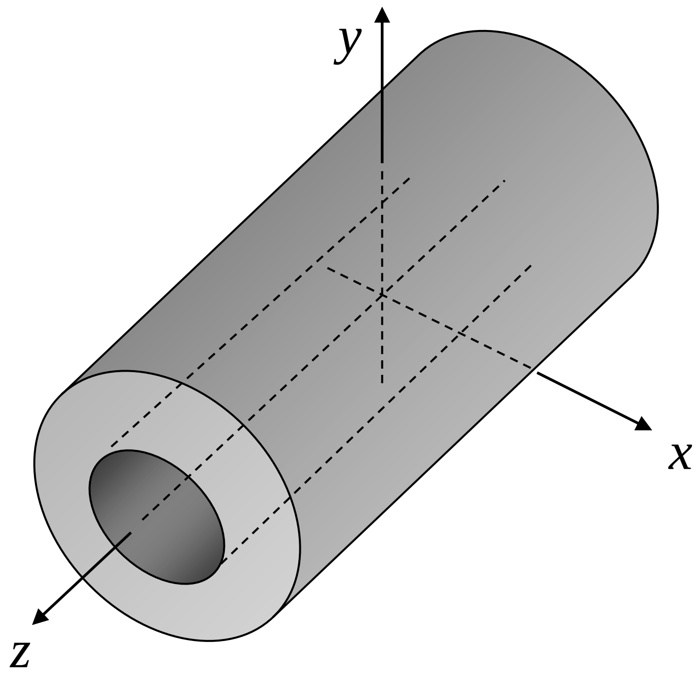
2D Constitutive Law: $\frac{\partial^2}{\partial y^2}\left(\sigma_x - \nu(\sigma_x + \sigma_y)\right) + \frac{\partial^2}{\partial x^2}\left(\sigma_y - \nu(\sigma_x + \sigma_y)\right) = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$

Add $\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right)$ to both sides: $\nabla^2 (1-\nu)(\sigma_x + \sigma_y) = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$

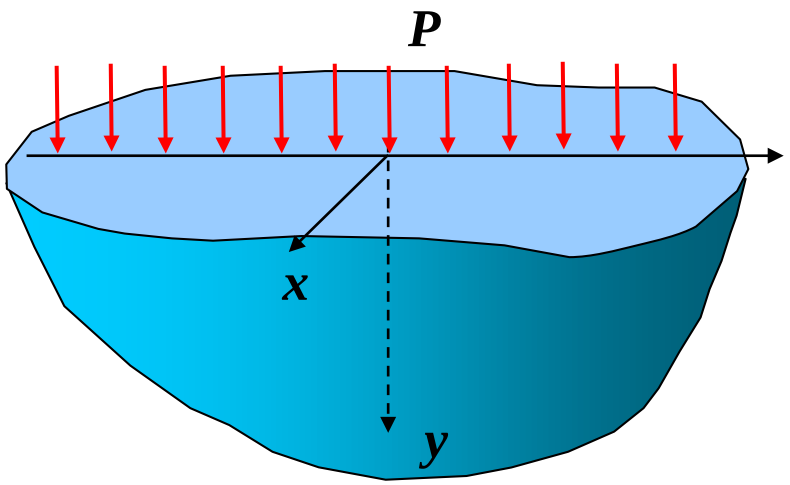
Using Equilibrium on the RHS: $\nabla^2 (1-\nu)(\sigma_x + \sigma_y) = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$

$$\Rightarrow \boxed{\nabla^2 (\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)}$$

Examples of Plane Strain Problems



Long cylinders
under uniform loading



Semi-infinite regions under
uniform loadings

Anti-Plane Strain

- An additional plane strain theory of elasticity called Anti-Plane Strain involves a formulation based on the existence of only out-of-plane deformation starting with an assumed displacement field: $u = v = 0,\ w = w(x, y).$

Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{xy} = 0,$$

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial w}{\partial x},\ \varepsilon_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}.$$

Stresses

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0,$$

$$\tau_{xz} = 2G\varepsilon_{xz},\ \tau_{yz} = 2G\varepsilon_{yz}.$$

Equilibrium Equations

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0,$$

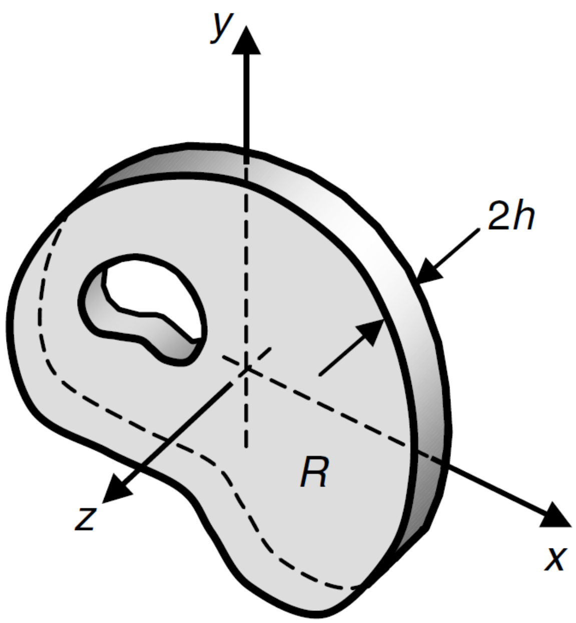
$$F_x = F_y = 0.$$

Navier's Equation

$$G\nabla^2 w + F_z = 0.$$

Plane Stress

- Consider the domain bounded two stress-free planes $z=\pm h$, where h is small in comparison with other dimensions in the problem.
- Since the region is thin in the z-direction, there can be little variation in the stress components σ_z, τ_{zx}, τ_{zy} through the thickness, and thus they will be approximately zero throughout the entire domain.
- Finally since the region is thin in the z-direction it can be argued that the other non-zero stresses will have little variation with z.
- Under these assumptions, the stress field can be simplified as



$$\sigma_x = \sigma_x(x, y)$$

$$\sigma_y = \sigma_y(x, y)$$

$$\tau_{xy} = \tau_{xy}(x, y)$$

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

Plane Stress Field Equations

- Isotropic Hooke's Law: $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij};\ \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy}, \quad \varepsilon_{zx} = \varepsilon_{zy} = 0$$

$$\sigma_x = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x,$$

$$\sigma_y = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y,$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$$

- Displacement-strain relation: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \ \varepsilon_y = \frac{\partial v}{\partial y}, \ \varepsilon_z = \frac{\partial w}{\partial z}, \ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \ \varepsilon_{yz} = 0, \ \varepsilon_{xz} = 0$$

Plane Stress Field Equations

- Equilibrium Equations

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0, \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0, \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0.
\end{aligned}
\Rightarrow
\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0, \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0.
\end{aligned}$$

Plane Stress Field Equations

- Navier's Equations

$$\sigma_x = \frac{E\nu}{(1+\nu)(1-\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{E}{(1+\nu)}\frac{\partial u}{\partial x},$$

$$\sigma_y = \frac{E\nu}{(1+\nu)(1-\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{E}{(1+\nu)}\frac{\partial v}{\partial y},$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+F_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+F_y = 0.$$

$$\Rightarrow \quad G\nabla^2 u+\frac{G(1+\nu)}{1-\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0,$$

$$G\nabla^2 v+\frac{G(1+\nu)}{1-\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y = 0.$$

$$G\nabla^2 \boldsymbol{u}+\frac{G(1+\nu)}{1-\nu}\nabla(\nabla\cdot\boldsymbol{u})+\boldsymbol{F} = 0.$$

Plane Stress Field Equations

- Strain Compatibility $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$ (6 eqns)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

- Beltrami-Michell Equation:

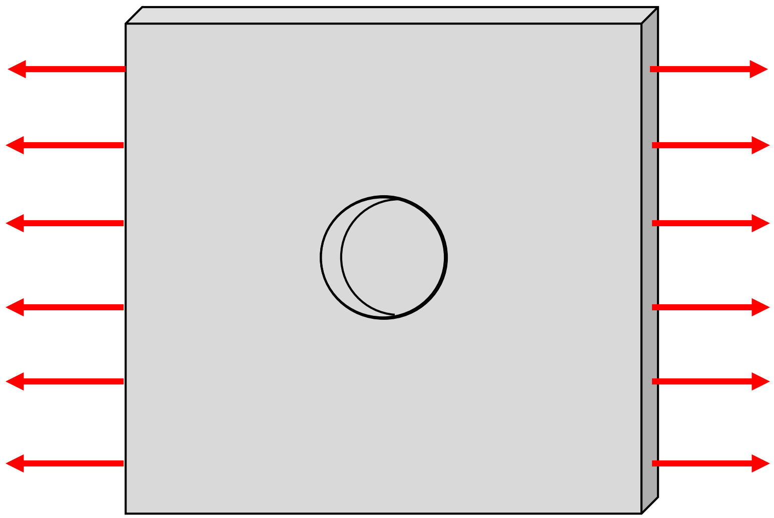
2D Constitutive Law: $\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$

Add $(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right)$ to both sides:

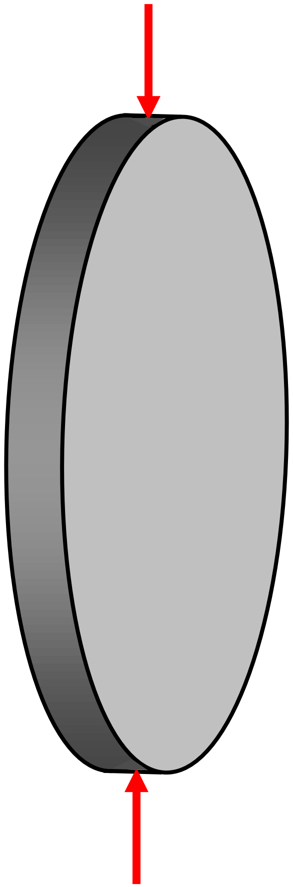
$$\nabla^2(\sigma_x + \sigma_y) = (1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}\right)$$

Using Equilibrium on the RHS: $\boxed{\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)}$

Examples of Plane Stress Problems



Thin plate with
central hole



Circular plate under
edge loadings

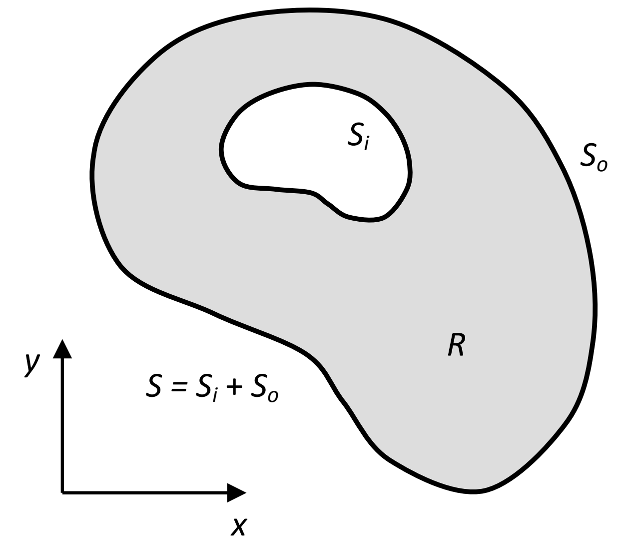
Plane Elasticity Boundary Value Problem

- **Displacement Boundary Conditions**

$$u = u_b(x,y)\ ,\ v = v_b(x,y)\ \text{ on } S_u$$

- **Stress/Traction Boundary Conditions**

$$\left.\begin{aligned} T_x^n &= T_x^{(b)}(x,y) = \sigma_x^{(b)} n_x + \tau_{xy}^{(b)} n_y \\ T_y^n &= T_y^{(b)}(x,y) = \tau_{xy}^{(b)} n_x + \sigma_y^{(b)} n_y \end{aligned}\right\} \text{ on } S_t$$



- **Plane Strain Problem:** Determine in-plane displacements, strains and stresses $\{u, v, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}, \sigma_x, \sigma_y, \tau_{xy}\}$ in R. Out-of-plane stress σ_z can be determined from in-plane stresses.

- **Plane Stress Problem:** Determine in-plane displacements, strains and stresses $\{u, v, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}, \sigma_x, \sigma_y, \tau_{xy}\}$ in R. Out-of-plane strain ε_z can be determined from in-plane strains.

Correspondence Between Plane Formulations

- Plane strain and plane stress field equations had identical equilibrium equations and boundary conditions.
- Navier's equations and compatibility relations were similar but not identical with differences occurring only in particular coefficients involving just elastic constants.
- So perhaps a simple change in elastic moduli would bring one set of relations into an exact match with the corresponding result from the other plane theory.

Correspondence Between Plane Formulations

Plane Strain

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_x + \nu\varepsilon_y\right),$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_y + \nu\varepsilon_x\right),$$

$$\tau_{xy} = \frac{E}{(1+\nu)}\varepsilon_{xy};$$

$$\varepsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right),$$

$$\varepsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right), \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy};$$

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right);$$

$$G\nabla^2 u + \frac{G}{(1-2\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0,$$

$$G\nabla^2 v + \frac{G}{(1-2\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0.$$

$$E \rightarrow \frac{E(1+2\nu)}{(1+\nu)^2}$$

$$\nu \rightarrow \frac{\nu}{1+\nu}$$

$$\frac{E}{1-\nu^2} \leftarrow E$$

$$\frac{\nu}{1-\nu} \leftarrow \nu$$

Plane Stress

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right), \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy},$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right);$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right),$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right), \quad \tau_{xy} = 2G\varepsilon_{xy};$$

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right);$$

$$G\nabla^2 u + \frac{G(1+\nu)}{1-\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0,$$

$$G\nabla^2 v + \frac{G(1+\nu)}{1-\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0.$$

- Thus, we only need to derive one set of equations: either plane strain or plane stress equations in 2D elasticity.

Combined Plane Formulations

- Define Kolosov's constant κ that is related to v

For plane strain: $\kappa = 3 - 4\nu$ or $\nu = \dfrac{3-\kappa}{4}$;

For plane stress: $\kappa = \dfrac{3-\nu}{1+\nu}$ or $\nu = \dfrac{3-\kappa}{1+\kappa}$.

- Constitutive relations:

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\sigma_{\alpha\beta} - \frac{3-\kappa}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\sigma_{\alpha\beta} = 2G\left(\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\varepsilon_x = \frac{1}{2G}\frac{1+\kappa}{4}\left(\sigma_x - \frac{3-\kappa}{1+\kappa}\sigma_y\right), \varepsilon_y = \frac{1}{2G}\frac{1+\kappa}{4}\left(\sigma_y - \frac{3-\kappa}{1+\kappa}\sigma_x\right), \varepsilon_{xy} = \frac{1}{2G}\tau_{xy}$$

$$\sigma_x = -\frac{G}{1-\kappa}\left((1+\kappa)\varepsilon_x + (3-\kappa)\varepsilon_y\right), \sigma_y = -\frac{G}{1-\kappa}\left((1+\kappa)\varepsilon_y + (3-\kappa)\varepsilon_x\right), \tau_{xy} = 2G\varepsilon_{xy}.$$

$$\varepsilon_{\gamma\gamma} = -\frac{1}{2G}\frac{1-\kappa}{2}\sigma_{\gamma\gamma}; \quad \sigma_{\gamma\gamma} = -2G\frac{2}{1-\kappa}\varepsilon_{\gamma\gamma}$$

Combined Plane Formulations

- Beltrami-Michell Equation:

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{4}{1+\kappa}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

- Navier's equations

$$\begin{cases} \sigma_x = -\dfrac{G}{1-\kappa}\left((1+\kappa)\dfrac{\partial u}{\partial x} + (3-\kappa)\dfrac{\partial v}{\partial y}\right); \\ \sigma_y = -\dfrac{G}{1-\kappa}\left((1+\kappa)\dfrac{\partial v}{\partial y} + (3-\kappa)\dfrac{\partial u}{\partial x}\right); \tau_{xy} = G\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right); \end{cases} \qquad \begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + F_x = 0, \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + F_y = 0. \end{cases}$$

$$\Rightarrow \begin{array}{l} G\nabla^2 u - \dfrac{2G}{1-\kappa}\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) + F_x = 0, \\ G\nabla^2 v - \dfrac{2G}{1-\kappa}\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) + F_y = 0. \end{array}$$

$$G\nabla^2 \boldsymbol{u} - \frac{2G}{1-\kappa}\nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{F} = 0.$$

Airy Stress Function Method

- Numerous solutions to plane strain and plane stress problems can be determined using an Airy Stress Function technique.
- The method will reduce the general formulation to a single governing equation in terms of a single unknown.
- The resulting equation is then solvable by several methods of applied mathematics, and thus many analytical solutions to problems of interest can be found.

Conservative Body Forces

- If a force field is capable of being represented as the gradient of a scalar function, it is referred to as conservative:

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

- Consider the work done when moving one particle in a gravitational field. The conservation of energy demands

$$\int_C dV = -\int_C \boldsymbol{F} \cdot d\boldsymbol{r} \quad \Rightarrow \quad \int_C \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = -\int_C \left(F_x dx + F_y dy + F_z dz \right)$$

- The curl: $\nabla \times \boldsymbol{F} = \varepsilon_{ijk} F_{k,j} \boldsymbol{e}_i = -\varepsilon_{ijk} \frac{\partial^2 V}{\partial x_k \partial x_j} \boldsymbol{e}_i = \boldsymbol{0}$

- Conservative force fields are irrotational. The above relation serves as the constraint condition.

Particular Cases of Conservative Body Forces

- Gravitational Loading

$$F_x = 0, \quad F_y = -\rho g \quad \Rightarrow \quad V = \rho g y \quad \& \quad \nabla^2 V = 0$$

- Inertial forces due to a constant angular velocity ω

$$a_r = \omega^2 r \quad \Rightarrow \quad a_x = \omega^2 x, \quad a_y = \omega^2 y$$

$$F_x = \rho a_x = \rho \omega^2 x, \quad F_y = \rho a_y = \rho \omega^2 y$$

$$\Rightarrow V = -\frac{1}{2} \rho \omega^2 \left(x^2 + y^2 \right) \quad \Rightarrow \quad \nabla^2 V = -2 \rho \omega^2$$

- Inertial forces due to rigid-body accelerations are conservative if and only if angular velocity is constant.

Airy Stress Function Method

- Equilibrium equations for plane problems

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0.$$

- **In the case of a body force derivable from a potential function, i.e. a conservative body force**

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}$$

- Solution to the homogeneous equations

$$\frac{\partial (\sigma_x - V)}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial (\sigma_y - V)}{\partial y} = 0.$$

Airy Stress Function Method

- By the theory of differential equations

$$\begin{cases} \dfrac{\partial(\sigma_x - V)}{\partial x} = -\dfrac{\partial \tau_{xy}(x,y)}{\partial y} \\ \dfrac{\partial \tau_{xy}(x,y)}{\partial x} = -\dfrac{\partial(\sigma_y - V)}{\partial y} \end{cases} \Rightarrow \begin{cases} \sigma_x - V = \dfrac{\partial A(x,y)}{\partial y}, \quad -\tau_{xy} = \dfrac{\partial A(x,y)}{\partial x} \\ -\tau_{xy} = \dfrac{\partial B(x,y)}{\partial y}, \quad \sigma_y - V = \dfrac{\partial B(x,y)}{\partial x} \end{cases}$$

$$\Rightarrow \frac{\partial A(x,y)}{\partial x} = \frac{\partial B(x,y)}{\partial y} \Rightarrow A(x,y) = \frac{\partial \psi(x,y)}{\partial y}, \quad B(x,y) = \frac{\partial \psi(x,y)}{\partial x}$$

$$\Rightarrow \boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2} + V, \sigma_y = \frac{\partial^2 \psi}{\partial x^2} + V, \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y},}$$

- where $\psi = \psi(x,y)$ is an arbitrary form called *Airy's Stress Function*. This stress form automatically satisfies the equilibrium equation.

Airy Stress Function Method

- Beltrami-Michell Equation

$$\nabla^2(\sigma_x+\sigma_y) = -\frac{4}{1+\kappa}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \Rightarrow \boxed{\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=\nabla^4\psi=\frac{2(1-\kappa)}{1+\kappa}\nabla^2 V}$$

$$\text{Plane strain: } \kappa = 3-4\nu, \quad \text{Plane stress: } \kappa = \frac{3-\nu}{1+\nu}.$$

- **For harmonic body force potentials, i.e. gravity**

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=\nabla^4\psi=0.$$

- This relation is called the *biharmonic equation* and its solutions are known as *biharmonic functions*.
- The governing Airy stress function equation is identical for plane strain and plane stress, and is independent of elastic constants.
- If only traction BCs are specified for a simply connected region, the stress field for both cases is also identical.

Airy Stress Function Formulation

- The plane problem of elasticity can be reduced to a single equation in terms of the Airy stress function.
- Traction boundary conditions would involve the specification of second derivatives of the stress function; however, this condition can be reduced to specification of first order derivatives.

$$T_x^{(n)} = \sigma_x n_x + \tau_{xy} n_y = \frac{\partial^2 \psi}{\partial y^2} n_x - \frac{\partial^2 \psi}{\partial x \partial y} n_y ,$$

$$T_y^{(n)} = \tau_{xy} n_x + \sigma_y n_y = -\frac{\partial^2 \psi}{\partial x \partial y} n_x + \frac{\partial^2 \psi}{\partial x^2} n_y .$$

- The plane problem is then formulated in terms of an Airy function with a single governing biharmonic equation.

Polar Coordinate Formulation

- Strain-Displacement relationship

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right).$$

- Hooke's Law

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\sigma_{\alpha\beta} - \frac{3-\kappa}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\sigma_{\alpha\beta} = 2G\left(\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\varepsilon_r = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_r - \frac{3-\kappa}{1+\kappa}\sigma_\theta\right), \varepsilon_\theta = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_\theta - \frac{3-\kappa}{1+\kappa}\sigma_r\right), \varepsilon_{r\theta} = \frac{1}{2G}\tau_{r\theta}.$$

$$\sigma_r = -\frac{G}{(1-\kappa)}\left((1+\kappa)\varepsilon_r + (3-\kappa)\varepsilon_\theta\right), \sigma_\theta = -\frac{G}{(1-\kappa)}\left((1+\kappa)\varepsilon_\theta + (3-\kappa)\varepsilon_r\right), \tau_{r\theta} = 2G\varepsilon_{r\theta}.$$

$$\text{For plane strain: } \kappa = 3 - 4\nu; \quad \text{For plane stress: } \kappa = \frac{3-\nu}{1+\nu}.$$

Polar Coordinate Formulation

- Equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0, \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0.$$

- Navier's equation

$$G\nabla^2 \boldsymbol{u} - \frac{2G}{1-\kappa}\nabla(\nabla\cdot\boldsymbol{u}) + \boldsymbol{F} = 0.$$

$$\Rightarrow \begin{cases} G\left(\dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial \theta^2} - \dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial \theta} - \dfrac{u_r}{r^2}\right) - \dfrac{2G}{1-\kappa}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right) + F_r = 0, \\ G\left(\dfrac{\partial^2 u_\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r^2}\right) - \dfrac{2G}{1-\kappa}\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right) + F_\theta = 0. \end{cases}$$

- Beltrami-Michell equation

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{4}{1+\kappa}\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right).$$

Airy Stress Function in Polar Coordinates

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial y^2} + V, \quad \sigma_{22} = \frac{\partial^2 \psi}{\partial x^2} + V, \quad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x \partial y}.$$

$$\begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{Bmatrix}$$

$$\sigma'_{ij} = Q_{ik} Q_{jl} \sigma_{kl} = Q_{i1} Q_{j1} \sigma_{11} + Q_{i1} Q_{j2} \sigma_{12} + Q_{i2} Q_{j1} \sigma_{21} + Q_{i2} Q_{j2} \sigma_{22}$$

$$\sigma'_{11} = Q_{11} Q_{11} \sigma_{11} + Q_{11} Q_{12} \sigma_{12} + Q_{12} Q_{11} \sigma_{21} + Q_{12} Q_{12} \sigma_{22} = \cos^2\theta \frac{\partial^2 \psi}{\partial y^2} - 2 \sin\theta \cos\theta \frac{\partial^2 \psi}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 \psi}{\partial x^2}$$

$$\sigma'_{22} = Q_{21} Q_{21} \sigma_{11} + Q_{21} Q_{22} \sigma_{12} + Q_{22} Q_{21} \sigma_{21} + Q_{22} Q_{22} \sigma_{22} = \sin^2\theta \frac{\partial^2 \psi}{\partial y^2} + 2 \sin\theta \cos\theta \frac{\partial^2 \psi}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 \psi}{\partial x^2}$$

$$\sigma_{12} = Q_{11} Q_{21} \sigma_{11} + Q_{11} Q_{22} \sigma_{12} + Q_{12} Q_{21} \sigma_{21} + Q_{12} Q_{22} \sigma_{22} = -\sin\theta \cos\theta \frac{\partial^2 \psi}{\partial y^2} - \left(\cos^2\theta - \sin^2\theta\right) \frac{\partial^2 \psi}{\partial x \partial y} + \sin\theta \cos\theta \frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow \boxed{\sigma_r = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + V, \quad \sigma_\theta = \frac{\partial^2 \psi}{\partial r^2} + V, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial \psi}{\partial \theta}\right).}$$

Airy Stress Function in Polar Coordinates

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\cos\theta + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$= \sin\theta\cos\theta \frac{\partial^2}{\partial r^2} - \frac{\cos^2\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial^2}{\partial r \partial \theta}$$

$$- \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r} - \frac{\sin^2\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Airy Stress Function in Polar Coordinates

- Beltrami-Michell equation

$$\nabla^4 \psi = \frac{2(1-\kappa)}{1+\kappa} \nabla^2 V \quad \Rightarrow$$

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi = \frac{2(1-\kappa)}{1+\kappa}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)V}$$

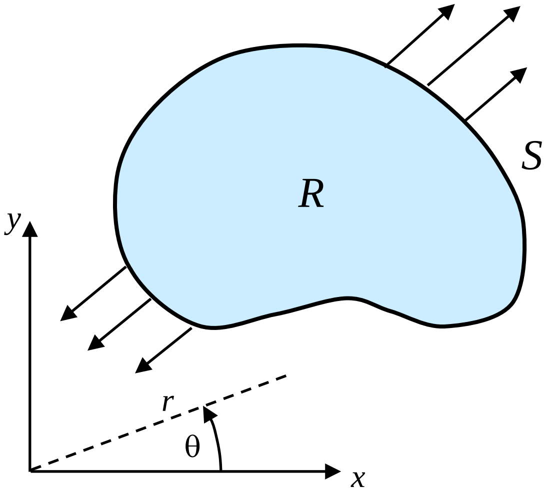
- Traction boundary conditions

$$f_r(r,\theta) = T_r^{(n)} = \sigma_r n_r + \tau_{r\theta} n_\theta$$

$$= \left(\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) n_r - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) n_\theta ,$$

$$f_\theta(r,\theta) = T_\theta^{(n)} = \tau_{r\theta} n_r + \sigma_\theta n_\theta$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) n_r + \frac{\partial^2 \psi}{\partial r^2} n_\theta .$$



- The plane problem is then formulated in terms of an Airy function with a single governing biharmonic equation.

Outline

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- Plane Strain
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