Two-Dimensional Problems in Polar Coordinates

Outline

- Polar Coordinate Formulation
- Axisymmetric Solutions to Biharmonic Equations
- Cylinders under Boundary Pressures
- Hole in Infinite Media
- Pure Bending of Curved Beams
- Rotating Disk/Cylinder Problem
- General Solutions to Biharmonic equation
- Stress Concentration around a Hole
- Transverse Bending of Curved Beams
- Wedge Problems
- Quarter-Plane Problems
- Half-Plane Problems

Polar Coordinate Formulation – Review

- Strain-Displacement

$$\boxed{\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right).}$$

- Hooke's Law

$$\boxed{\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\sigma_{\alpha\beta} - \frac{3-\kappa}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)} \quad \boxed{\sigma_{\alpha\beta} = 2G\left(\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right)}$$

$$\boxed{\varepsilon_r = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_r - \frac{3-\kappa}{1+\kappa}\sigma_\theta\right), \varepsilon_\theta = \frac{1}{2G}\frac{(1+\kappa)}{4}\left(\sigma_\theta - \frac{3-\kappa}{1+\kappa}\sigma_r\right), \varepsilon_{r\theta} = \frac{1}{2G}\tau_{r\theta}.}$$

$$\boxed{\sigma_r = -\frac{G}{(1-\kappa)}\left((1+\kappa)\varepsilon_r + (3-\kappa)\varepsilon_\theta\right), \sigma_\theta = -\frac{G}{(1-\kappa)}\left((1+\kappa)\varepsilon_\theta + (3-\kappa)\varepsilon_r\right), \tau_{r\theta} = 2G\varepsilon_{r\theta}.}$$

$$\boxed{\text{For plane strain: } \kappa = 3-4\nu; \quad \text{For plane stress: } \kappa = \frac{3-\nu}{1+\nu}.}$$

Polar Coordinate Formulation – Review

- Equilibrium equations

$$\boxed{\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\sigma_r-\sigma_\theta}{r}+F_r=0, \quad \frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{2\tau_{r\theta}}{r}+F_\theta=0.}$$

- Beltrami-Michell equation

$$\boxed{\nabla^2\left(\sigma_r+\sigma_\theta\right)=-\frac{4}{1+\kappa}\left(\frac{\partial F_r}{\partial r}+\frac{F_r}{r}+\frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right).}$$

- Navier's equation

$$G\nabla^2\boldsymbol{u}-\frac{2G}{1-\kappa}\nabla\left(\nabla\cdot\boldsymbol{u}\right)+\boldsymbol{F}=0.$$

$$\Rightarrow\begin{cases} G\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{u_r}{r^2}\right)-\frac{2G}{1-\kappa}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+F_r=0, \\ G\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2}\right)-\frac{2G}{1-\kappa}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+F_\theta=0. \end{cases}$$

Polar Coordinate Formulation – Review

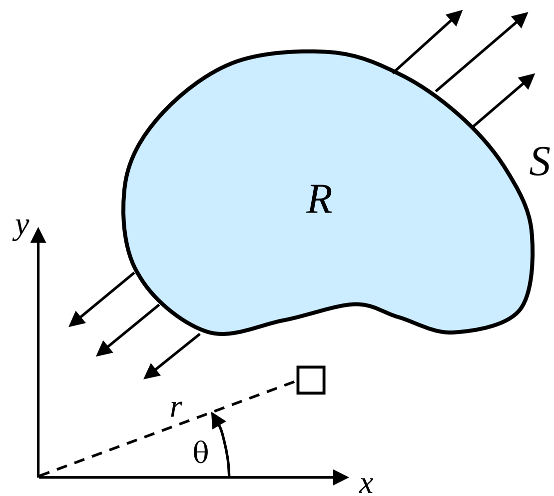
- Airy Stress Function representation

$$\boxed{\begin{gathered}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi=\frac{2(1-\kappa)}{1+\kappa}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)V.\\ \sigma_r=\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+V,\quad \sigma_\theta=\frac{\partial^2\psi}{\partial r^2}+V,\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right).\end{gathered}}$$

- Traction boundary conditions

$$f_r(r,\theta)=T_r^{(\boldsymbol{n})}=\sigma_r n_r+\tau_{r\theta}n_\theta$$

$$f_\theta(r,\theta)=T_\theta^{(\boldsymbol{n})}=\tau_{r\theta}n_r+\sigma_\theta n_\theta$$



- Without body forces, the plane problem is then reduced to a single governing biharmonic equation.

Axisymmetric Solutions

- Navier's **Displacement Formulation** (without body forces)

$$\boldsymbol{u} = u_r\left(r\right)\boldsymbol{e}_r, \quad \boldsymbol{F} = \boldsymbol{0}.$$

$$\begin{cases} G\left(\dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial r} + \cancel{\dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial \theta^2} - \dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial \theta}} - \dfrac{u_r}{r^2}\right) - \dfrac{2G}{1-\kappa}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \cancel{\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}}\right) + \cancel{F_r} = 0, \\ \cancel{G\left(\dfrac{\partial^2 u_\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r^2}\right) - \dfrac{2G}{1-\kappa}\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right) + F_\theta = 0.} \end{cases}$$

- Displacement field

$$\Rightarrow \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = 0 \;\Rightarrow\; \frac{du_r}{dr} + \frac{u_r}{r} = C_1 \;\Rightarrow\; \frac{du_r}{dr} + \frac{u_r}{r} = 2C_1 \Rightarrow \boxed{u_r = C_1 r + C_2\frac{1}{r}}$$

- Strain and stress field

$$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} = C_1 - C_2\dfrac{1}{r^2}, \\ \varepsilon_\theta = \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta}\right) = C_1 + C_2\dfrac{1}{r^2}, \\ \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right) = 0. \end{cases} \Rightarrow \begin{cases} \sigma_r = -\dfrac{G}{\left(1-\kappa\right)}\left(\left(1+\kappa\right)\varepsilon_r + \left(3-\kappa\right)\varepsilon_\theta\right) = -2G\left(\dfrac{2}{1-\kappa}C_1 + C_2\dfrac{1}{r^2}\right), \\ \sigma_\theta = -\dfrac{G}{\left(1-\kappa\right)}\left(\left(1+\kappa\right)\varepsilon_\theta + \left(3-\kappa\right)\varepsilon_r\right) = -2G\left(\dfrac{2}{1-\kappa}C_1 - C_2\dfrac{1}{r^2}\right), \\ \tau_{r\theta} = 2G\varepsilon_{r\theta} = 0. \end{cases} \Rightarrow \boxed{\begin{array}{l} \sigma_r = \dfrac{A}{r^2} + B, \\ \sigma_\theta = -\dfrac{A}{r^2} + B, \\ \tau_{r\theta} = 0. \end{array}}$$

Axisymmetric Solutions

- Airy **Stress Formulation** (without body forces)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

$$\Rightarrow \nabla^4\psi = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right]\right\} = 0$$

$$\Rightarrow r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right] = A \Rightarrow \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right] = \frac{A}{r} \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = A\ln r + B$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = Ar\ln r + Br \Rightarrow r\frac{d\psi}{dr} = A_1 r^2\ln r + B_1 r^2 + C$$

$$\Rightarrow \frac{d\psi}{dr} = A_1 r\ln r + B_1 r + \frac{C}{r} \Rightarrow \psi = A_2 r^2\ln r + B_2 r^2 + C\ln r + D$$

$$\Rightarrow \boxed{\psi = a_0 + a_1\ln r + a_2 r^2 + a_3 r^2\ln r}$$

Axisymmetric Solutions

- Stress and strain field

$$\psi = a_0 + a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r$$

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + \cancel{\dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2}}, \\ \sigma_\theta = \dfrac{\partial^2 \psi}{\partial r^2}, \\ \tau_{r\theta} = \cancel{-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}\right)}. \end{cases} \Rightarrow \begin{cases} \sigma_r = \dfrac{a_1}{r^2} + 2a_2 + a_3\left(1 + 2\ln r\right) \\ \sigma_\theta = -\dfrac{a_1}{r^2} + 2a_2 + a_3\left(3 + 2\ln r\right) \\ \tau_{r\theta} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon_r = \dfrac{1}{2G}\dfrac{(1+\kappa)}{4}\left(\sigma_r - \dfrac{3-\kappa}{1+\kappa}\sigma_\theta\right) = \dfrac{1}{2G}\dfrac{(1+\kappa)}{4}\left(\dfrac{4}{1+\kappa}\dfrac{a_1}{r^2} - \dfrac{4(1-\kappa)}{1+\kappa}a_2 - a_3\left(\dfrac{4(2-\kappa)}{1+\kappa} + \dfrac{4(1-\kappa)}{1+\kappa}\ln r\right)\right), \\ \varepsilon_\theta = \dfrac{1}{2G}\dfrac{(1+\kappa)}{4}\left(\sigma_\theta - \dfrac{3-\kappa}{1+\kappa}\sigma_r\right) = \dfrac{1}{2G}\dfrac{(1+\kappa)}{4}\left(-\dfrac{4}{1+\kappa}\dfrac{a_1}{r^2} - \dfrac{4(1-\kappa)}{1+\kappa}a_2 + a_3\left(\dfrac{4\kappa}{1+\kappa} - \dfrac{4(1-\kappa)}{1+\kappa}\ln r\right)\right), \\ \varepsilon_{r\theta} = \dfrac{1}{2G}\tau_{r\theta} = 0. \end{cases}$$

Axisymmetric Solutions

- Displacement field

$$u_r = \int \varepsilon_r dr = -\frac{1}{2G}\left(\frac{a_1}{r} + (1-\kappa)a_2 r + a_3\left(r + (1-\kappa) r \ln r\right)\right) + f(\theta)$$

$$u_\theta = \int (r\varepsilon_\theta - u_r)\, d\theta = \int \left(\frac{1+\kappa}{2G} a_3 r - f(\theta)\right) d\theta = \frac{1+\kappa}{2G} a_3 r\theta - \int f(\theta)\, d\theta + g(r)$$

$$0 = \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \Rightarrow \; f'(\theta) + \int f(\theta)\, d\theta - g(r) + r g'(r) = 0$$

$$\Rightarrow f'(\theta) + \int f(\theta)\, d\theta = g(r) - r g'(r) = K \;\; \Rightarrow \; g(r) = \omega_o r + K, \;\; f(\theta) = u_o \cos\theta + v_o \sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} u_r &= -\frac{1}{2G}\left(\frac{a_1}{r} + (1-\kappa)a_2 r + a_3\left(r + (1-\kappa) r \ln r\right)\right) + u_o \cos\theta + v_o \sin\theta \\ u_\theta &= \frac{1+\kappa}{2G} a_3 r\theta - u_o \sin\theta + v_o \cos\theta + \omega_o r \cancel{+K}, \qquad K \equiv 0 \end{aligned}}$$

- **Identification of rigid-body displacements in polar coordinates**

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} -\omega_o y + u_o \\ \omega_o x + v_o \end{Bmatrix} = \begin{Bmatrix} \cos\theta(-\omega_o r \sin\theta + u_o) + \sin\theta(\omega_o r\cos\theta + v_o) \\ -\sin\theta(-\omega_o r\sin\theta + u_o) + \cos\theta(\omega_o r \cos\theta + v_o) \end{Bmatrix} = \begin{Bmatrix} u_o \cos\theta + v_o \sin\theta \\ -u_o \sin\theta + v_o\cos\theta + \omega_o r \end{Bmatrix}$$

Axisymmetric Solutions

- Displacement formulation

$$\boxed{\begin{aligned} u_r &= C_1 r + C_2 \frac{1}{r}, \\ u_\theta &= 0. \quad (\text{assumed}) \end{aligned}}$$

$$\boxed{\begin{aligned} \sigma_r &= -2G\left(\frac{2}{1-\kappa}C_1 + C_2\frac{1}{r^2}\right), \\ \sigma_\theta &= -2G\left(\frac{2}{1-\kappa}C_1 - C_2\frac{1}{r^2}\right). \end{aligned}}$$

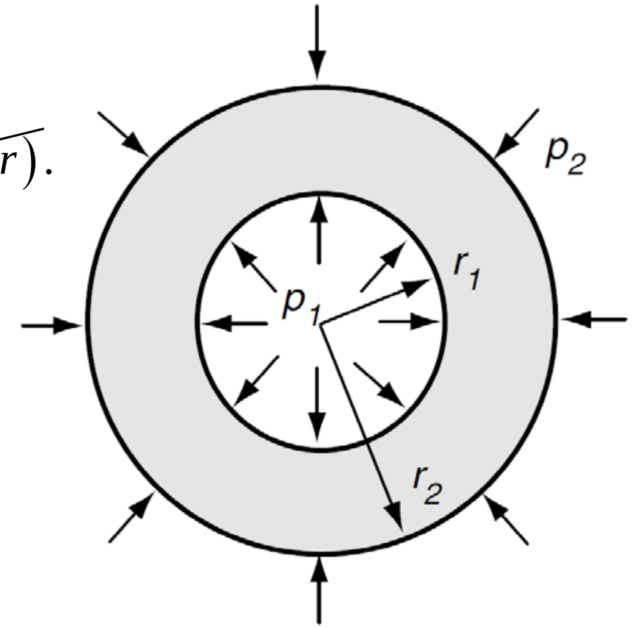
- Stress formulation

$$\boxed{\begin{aligned} \sigma_r &= \frac{a_1}{r^2} + 2a_2 + a_3(1+2\ln r), \\ \sigma_\theta &= -\frac{a_1}{r^2} + 2a_2 + a_3(3+2\ln r). \end{aligned}}$$

$$\boxed{\begin{aligned} u_r &= -\frac{1}{2G}\left(\begin{array}{l} \dfrac{a_1}{r} + (1-\kappa)a_2 r \\ +a_3\left(r + (1-\kappa) r \ln r\right) \end{array}\right) + u_o \cos\theta + v_o \sin\theta, \\ u_\theta &= \frac{1+\kappa}{2G} a_3 r\theta - u_o \sin\theta + v_o \cos\theta + \omega_o r. \end{aligned}}$$

- The displacement formulation does not contain the logarithmic terms. Thus, these terms are not consistent with single-valued displacements. The compatibility condition is not sufficient.
- The a_3 term leads to multivalued behavior, and is not found following the displacement formulation approach.
- The candidacy of individual terms depend on domain singularity.

Thick-Walled Cylinder Under Uniform Boundary Pressure



- General axisymmetric stress solution

$$\sigma_r = \frac{a_1}{r^2} + 2a_2 \cancel{+a_3\left(1+2\ln r\right)}, \quad \sigma_\theta = -\frac{a_1}{r^2} + 2a_2 \cancel{+a_3\left(3+2\ln r\right)}.$$

- Boundary conditions

$$\begin{cases} -p_1 = \sigma_r\left(r_1\right) = \dfrac{a_1}{r_1^2} + 2a_2 \\ -p_2 = \sigma_r\left(r_2\right) = \dfrac{a_1}{r_2^2} + 2a_2 \end{cases} \Rightarrow \begin{cases} a_1 = \dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - p_1\right) \\ 2a_2 = \dfrac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} \end{cases}$$

- Stresses

$$\sigma_r = \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - p_1\right)\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}, \quad \sigma_\theta = -\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - p_1\right)\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}.$$

- Displacements (depending on elastic constants)

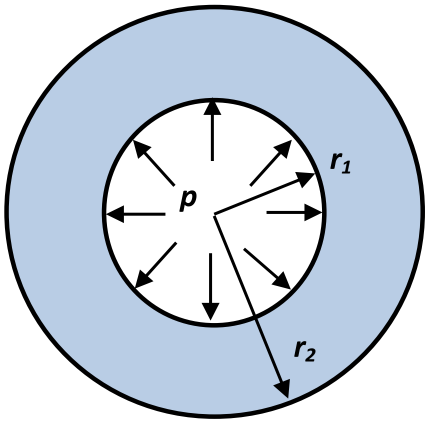
$$u_r = -\frac{1}{2G}\left(\frac{a_1}{r} + \left(1-\kappa\right)a_2 r \cancel{+a_3\left(r + \left(1-\kappa\right) r \ln r\right)}\right) = -\frac{1}{2G}\left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - p_1\right)\frac{1}{r} + \left(1-\kappa\right)\frac{r_1^2 p_1 - r_2^2 p_2}{2\left(r_2^2 - r_1^2\right)} r\right),$$

$$u_\theta = \cancel{\frac{1+\kappa}{2G} a_3 r\theta} = 0.$$

$\boxed{\text{For plane strain: } \kappa = 3 - 4\nu.}$

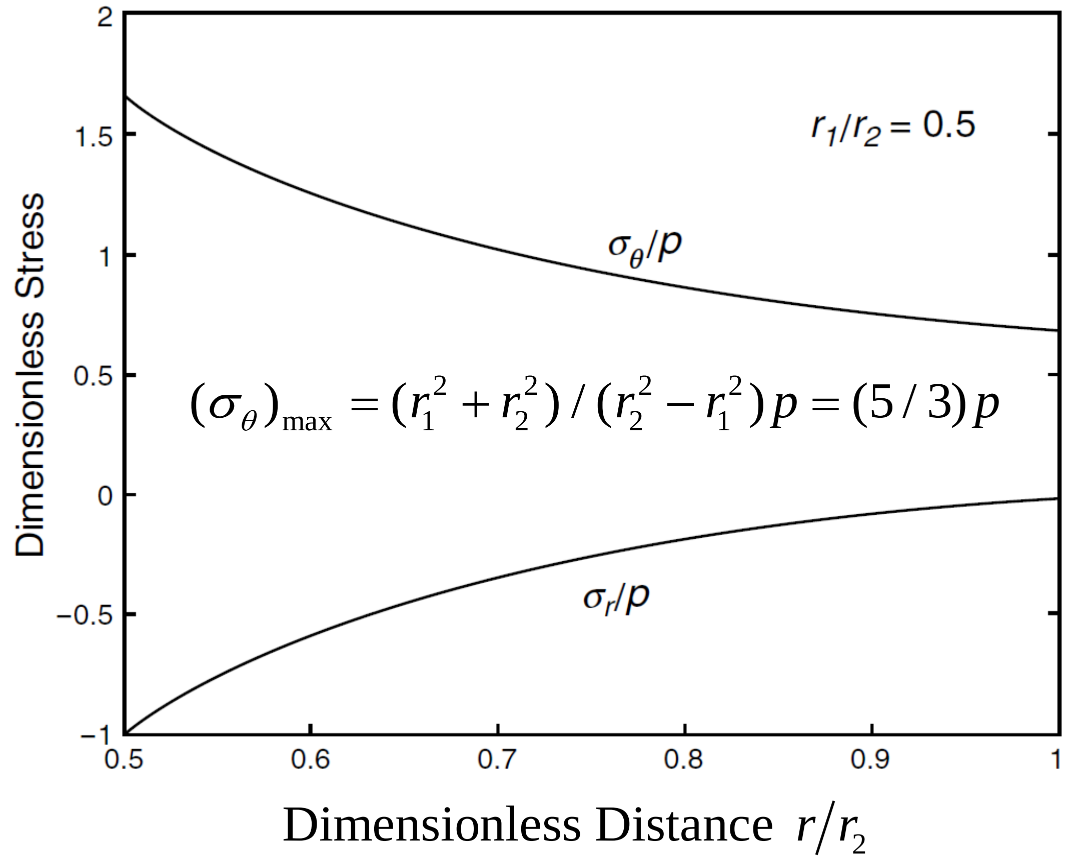
Thick-Walled Cylinder Under Internal Pressure

- Internal pressure only



$$\frac{\sigma_r}{p} = -\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2}{r_2^2 - r_1^2},$$

$$\frac{\sigma_\theta}{p} = \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2}{r_2^2 - r_1^2}.$$



- Thin-walled tubes

$$\begin{cases} r_o = (r_1 + r_2)/2 \\ t = r_2 - r_1 << r_o \end{cases} \Rightarrow \begin{cases} \sigma_r \approx 0, \\ \sigma_\theta \approx \dfrac{p r_0}{t}. \end{cases} \quad \text{(Matching with Strength of Materials Solution)}$$

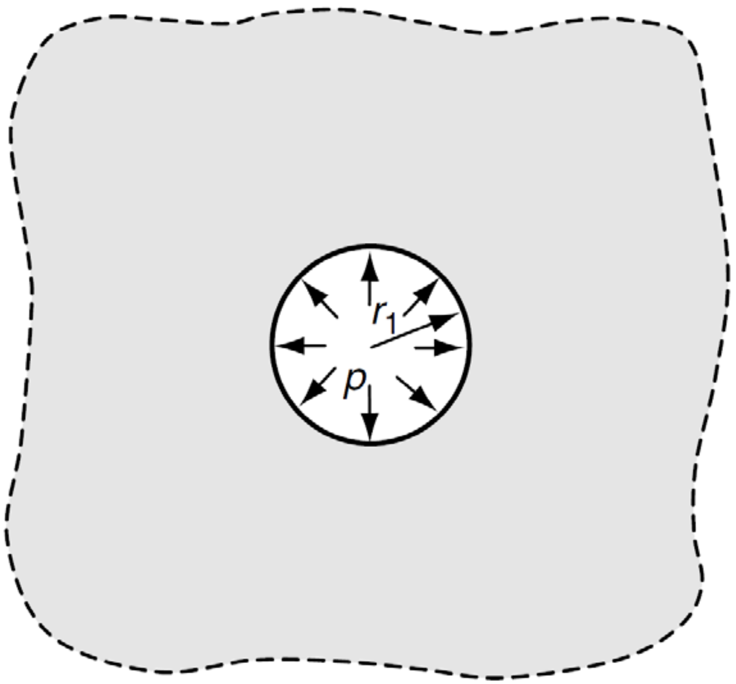
Hole in an Infinite Medium under Internal Pressure

- Internal pressure only and r tends to infinity

$$\boxed{p_2 = 0, \quad r_2 \to \infty}$$

$$\begin{cases} \sigma_r = \dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(\cancel{p_2} - p_1\right)\dfrac{1}{r^2} + \dfrac{r_1^2 p_1 \cancel{- r_2^2 p_2}}{r_2^2 - r_1^2}, \\ \sigma_\theta = -\dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(\cancel{p_2} - p_1\right)\dfrac{1}{r^2} + \dfrac{r_1^2 p_1 \cancel{- r_2^2 p_2}}{r_2^2 - r_1^2}. \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_r = -p_1 \dfrac{r_1^2}{r^2}, \\ \sigma_\theta = p_1 \dfrac{r_1^2}{r^2}, \end{cases}$$



$$u_r = -\frac{1}{2G}\left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(\cancel{p_2} - p_1\right)\frac{1}{r} + (1-\kappa)\frac{r_1^2 p_1 \cancel{- r_2^2 p_2}}{2\left(r_2^2 - r_1^2\right)} r\right)$$

$$\Rightarrow u_r = \frac{1}{2G} p_1 \frac{r_1^2}{r}.$$

Hole in an Infinite Medium under Biaxial Remote Tension

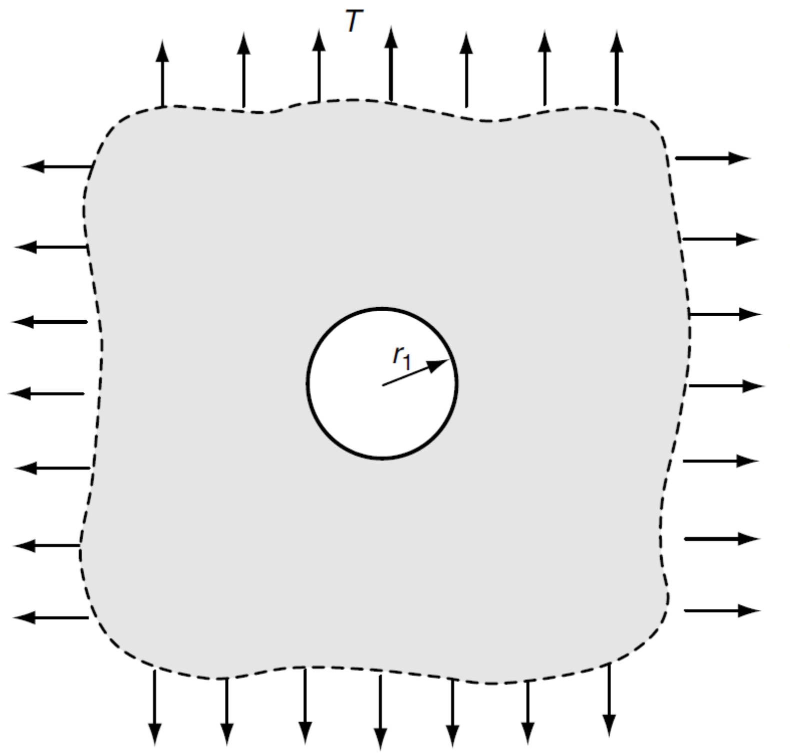
- Stress free hole in an infinite medium under remote biaxial loading

$$\boxed{p_1 = 0,\ p_2 = -T,\ r_2 \to \infty}$$

$$\begin{cases} \sigma_r = \dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - \cancel{p_1}\right)\dfrac{1}{r^2} + \dfrac{\cancel{r_1^2 p_1} - r_2^2 p_2}{r_2^2 - r_1^2}, \\ \sigma_\theta = -\dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - \cancel{p_1}\right)\dfrac{1}{r^2} + \dfrac{\cancel{r_1^2 p_1} - r_2^2 p_2}{r_2^2 - r_1^2}. \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_r = T\left(1 - \dfrac{r_1^2}{r^2}\right), \\ \sigma_\theta = T\left(1 + \dfrac{r_1^2}{r^2}\right). \end{cases}$$

$$\sigma_{\max} = (\sigma_\theta)_{\max} = \sigma_\theta(r_1) = 2T$$



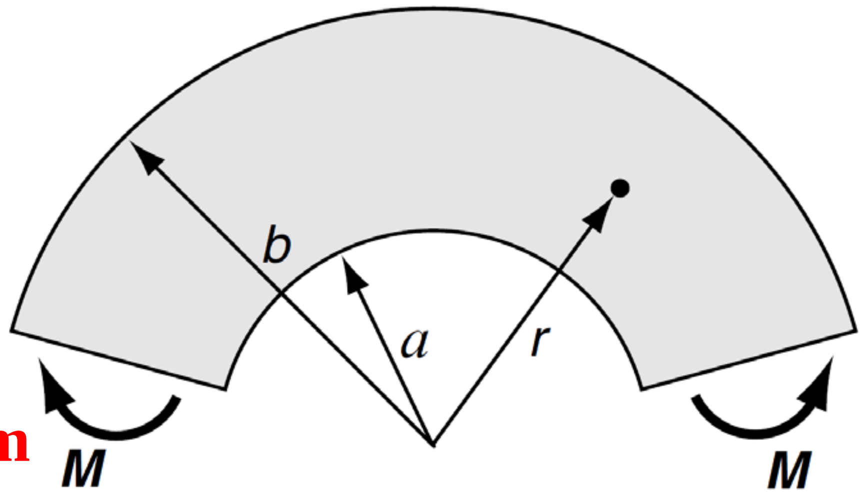
$$u_r = -\frac{1}{2G}\left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\left(p_2 - \cancel{p_1}\right)\frac{1}{r} + (1-\kappa)\frac{\cancel{r_1^2 p_1} - r_2^2 p_2}{2\left(r_2^2 - r_1^2\right)} r\right)$$

$$\Rightarrow u_r = -\frac{p_2}{2G}\left(\frac{r_1^2}{r} - \frac{1-\kappa}{2} r\right)$$

Pure Bending of Curved Beams

- Boundary conditions

$$\begin{cases} \sigma_r(a) = \sigma_r(b) = 0 \\ \tau_{r\theta}(a) = \tau_{r\theta}(b) = 0 \end{cases} \quad \begin{cases} \int_a^b \sigma_\theta dr = 0 \\ \int_a^b \sigma_\theta r dr = -M \end{cases}$$



- **This is an axisymmetric problem**

$$\psi = a_0 + a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r$$

$$\sigma_r = \frac{a_1}{r^2} + 2a_2 + a_3\left(1 + 2\ln r\right),\ \ \sigma_\theta = -\frac{a_1}{r^2} + 2a_2 + a_3\left(3 + 2\ln r\right),\ \ \tau_{r\theta} = 0$$

- Applying the BCs

$$a_1 = -\frac{4M}{N} a^2 b^2 \ln\frac{b}{a},\ \ a_2 = \frac{M}{N}\left(b^2 - a^2 + 2b^2 \ln b - 2a^2 \ln a\right),\ \ a_3 = -\frac{2M}{N}\left(b^2 - a^2\right),$$

where $N = \left(b^2 - a^2\right)^2 - 4a^2 b^2 \left(\ln\frac{b}{a}\right)^2$

Pure Bending of Curved Beams

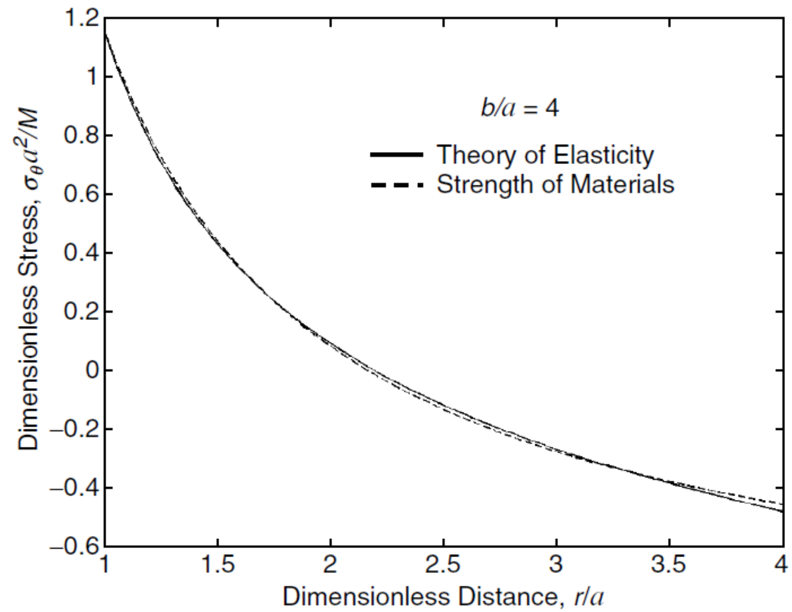
- Stress field

$$\sigma_r = -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r}\right),\ \sigma_\theta = -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r} + b^2 - a^2\right)$$

- Displacement field

$$u_r = -\frac{1}{2G}\begin{pmatrix}\frac{a_1}{r} + (1-\kappa)a_2 r \\ +a_3\left(r + (1-\kappa)r\ln r\right)\end{pmatrix} + u_o\cos\theta + v_o\sin\theta$$

$$u_\theta = \frac{1+\kappa}{2G}a_3 r\theta - u_o\sin\theta + v_o\cos\theta + \omega_o r$$



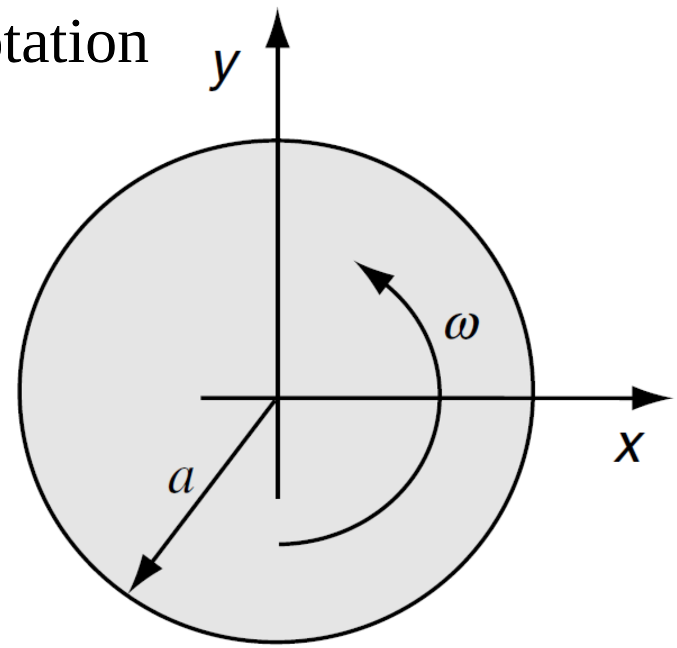
- Rotation of a polar differential element

$$\omega_{21} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\right) = \frac{1+\kappa}{2G}a_3\theta + \omega_o$$

Rotating Disk/Cylinder Problem

- Load: centrifugal force due to constant rotation

$$F_r = \rho\omega^2 r$$



- Equilibrium equations

$$\boxed{\frac{\partial\sigma_r}{\partial r} + \cancel{\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0,}$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\sigma_r\right) - \frac{\sigma_\theta}{r} + \rho\omega^2 r = 0$$

$$\Rightarrow \sigma_\theta = \frac{d}{dr}\left(r\sigma_r\right) + \rho\omega^2 r^2$$

- Propose a special stress function for this case

$$r\sigma_r = \psi, \quad \sigma_\theta = \frac{d\psi}{dr} + \rho\omega^2 r^2$$

- The equilibrium condition is automatically satisfied.

Rotating Disk/Cylinder Problem

- Beltrami-Michell equation

$$\nabla^2(\sigma_r+\sigma_\theta)=-\frac{4}{1+\kappa}\left(\frac{\partial F_r}{\partial r}+\frac{F_r}{r}\cancel{+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}}\right)\Rightarrow\nabla^2(\sigma_r+\sigma_\theta)=-\frac{8}{1+\kappa}\rho\omega^2$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\cancel{+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right),\ \sigma_r+\sigma_\theta=\frac{\psi}{r}+\frac{d\psi}{dr}+\rho\omega^2r^2=\frac{1}{r}\frac{d}{dr}(r\psi)+\rho\omega^2r^2$$

$$\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\frac{1}{r}\frac{d}{dr}(r\psi)+\rho\omega^2r^2\right)=-\frac{8}{1+\kappa}\rho\omega^2,\ \Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\psi)\right)\right)=-\frac{4(3+\kappa)}{1+\kappa}\rho\omega^2$$

$$\Rightarrow\psi=-\frac{3+\kappa}{4(1+\kappa)}\rho\omega^2r^3+\frac{1}{2}C_1r\left(\ln r-\frac{1}{2}\right)+\frac{1}{2}C_2r+\frac{C_3}{r}$$

$$\Rightarrow\begin{cases}\sigma_r=\dfrac{\psi}{r}=-\dfrac{3+\kappa}{4(1+\kappa)}\rho\omega^2r^2+\dfrac{1}{2}C_1\left(\ln r-\dfrac{1}{2}\right)+\dfrac{1}{2}C_2+\dfrac{C_3}{r^2}\\[2ex]\sigma_\theta=\dfrac{d\psi}{dr}+\rho\omega^2r^2=-\dfrac{5-\kappa}{4(1+\kappa)}\rho\omega^2r^2+\dfrac{1}{2}C_1\left(\ln r+\dfrac{1}{2}\right)+\dfrac{1}{2}C_2-\dfrac{C_3}{r^2}\end{cases}$$

- The C_1 term leads to multivalued displacement behavior, and is not found following the displacement formulation approach.

- Stress field $\begin{cases}\sigma_r(0,\theta),\ \sigma_\theta(0,\theta)\ \text{must be finite.}\\ \sigma_r(a,\theta)=0\end{cases}\Rightarrow\begin{cases}C_3=0\\ C_2=\dfrac{3+\kappa}{2(1+\kappa)}\rho\omega^2a^2\end{cases}$

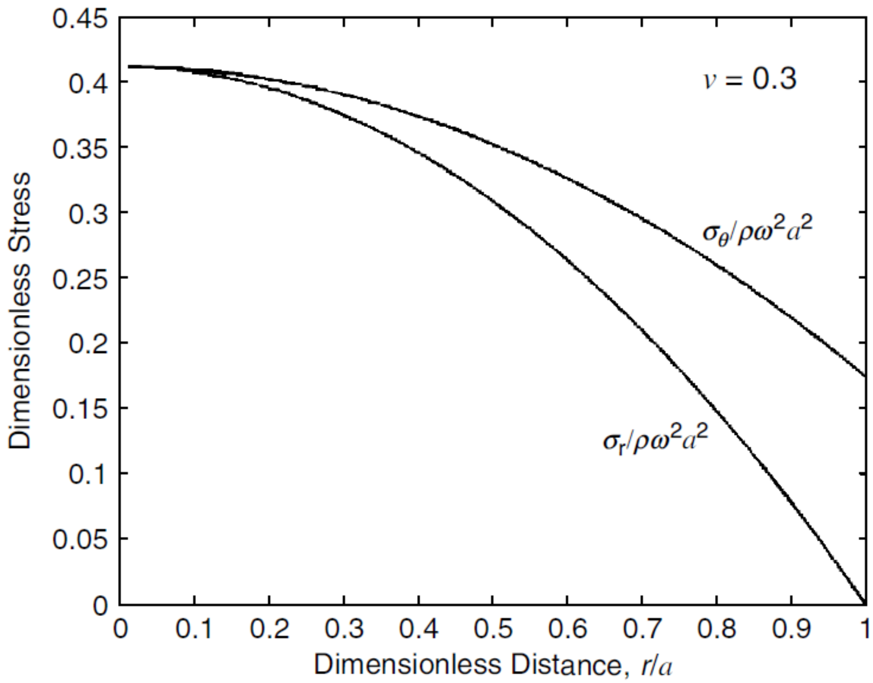
Rotating Disk/Cylinder Problem

- Stress field

$$\sigma_r = \frac{(3+\kappa)\rho\omega^2 a^2}{4(1+\kappa)}\left(1-\frac{r^2}{a^2}\right), \quad \sigma_\theta = \frac{\rho\omega^2 a^2}{4(1+\kappa)}\left((3+\kappa)-(5-\kappa)\frac{r^2}{a^2}\right)$$

- The maximum stress occurs at the center of the disk, even though the body force is largest at the outer boundary.

$$\sigma_{\max} = \sigma_r(0) = \sigma_\theta(0) = \frac{3+\kappa}{4(1+\kappa)}\rho\omega^2 a^2$$



- This problem can also be resolved in terms of the Navier's equation. Left as an exercise.

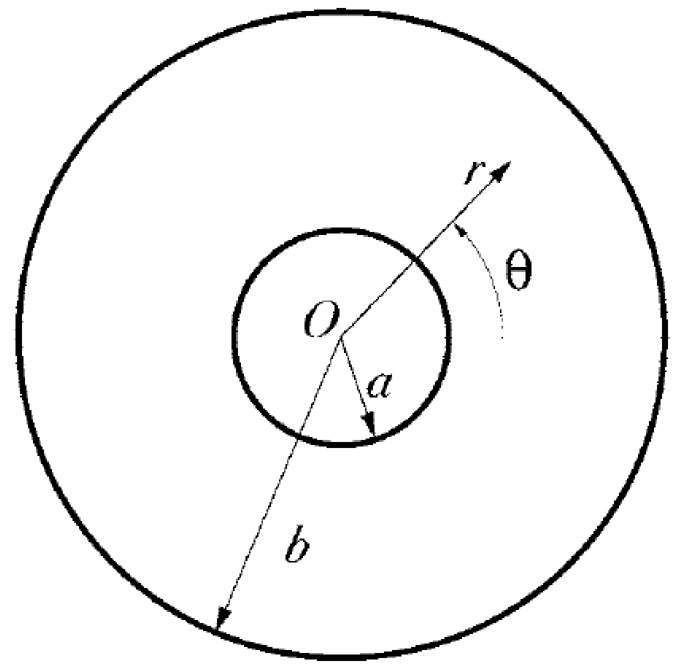
General Solution of Biharmonic Equation

- The most general case being the disk with a central hole, i.e. no θ-boundaries.
- The stresses and displacements must be single-valued and continuous and hence they must be periodic functions of θ.

$$\psi(r,\theta)=\sum_{n=0}^{\infty}\left\{ f_n(r)\cos n\theta+g_n(r)\sin n\theta\right\}$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi=0$$

$$\Rightarrow\begin{cases}\left(\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}-\dfrac{n^2}{r^2}\right)\left(\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}-\dfrac{n^2}{r^2}\right)f_n=0\\[2ex]\left(\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}-\dfrac{n^2}{r^2}\right)\left(\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}-\dfrac{n^2}{r^2}\right)g_n=0\end{cases}$$



$$\Rightarrow\begin{cases}f_n(r)=a_{n1}r^n+a_{n2}r^{2+n}+a_{n3}/r^n+a_{n4}/r^{n-2}, \qquad n\geq 2\\ f_0(r)=a_0+a_1\ln r+a_2r^2+a_3r^2\ln r, \qquad f_1(r)=a_{11}r+a_{12}r\ln r+a_{13}/r+a_{14}r^3\end{cases}$$

General Solution of Biharmonic Equation

- General solution with degeneracy

$$\psi(r,\theta)=\sum_{n=0}^{\infty}\left(f_n(r)\cos n\theta+g_n(r)\sin n\theta\right)$$

$$=\left(a_0+a_1\ln r+a_2r^2+a_3r^2\ln r\right)+\left(b_0+b_1\ln r+b_2r^2+b_3r^2\ln r\right)\sin\left((0)(\theta)\right)$$

$$+\left(a_{11}r+a_{12}r\ln r+a_{13}/r+a_{14}r^3\right)\cos\theta+\left(b_{11}r+b_{12}r\ln r+b_{13}/r+b_{14}r^3\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left(a_{n1}r^n+a_{n2}r^{2+n}+a_{n3}/r^n+a_{n4}/r^{n-2}\right)\cos n\theta+\sum_{n=2}^{\infty}\left(b_{n1}r^n+b_{n2}r^{2+n}+b_{n3}/r^n+b_{n4}/r^{n-2}\right)\sin n\theta$$

- Special treatment on degenerate terms

$$\psi=a_0=\lim_{\varepsilon\to 0}\left\{\frac{1}{2}a_0\left(r^{\varepsilon}+r^{-\varepsilon}\right)+\frac{1}{2}a_0'\left(r^{\varepsilon}-r^{-\varepsilon}\right)\right\}=\lim_{\varepsilon\to 0}\left\{\frac{1}{2}a_0\left(r^{\varepsilon}+r^{-\varepsilon}\right)+\frac{1}{2}a_0''\frac{r^{\varepsilon}-r^{-\varepsilon}}{\varepsilon}\right\}=a_0+a_0''\ln r$$

$$\psi=a_0=\lim_{\varepsilon\to 0}\left\{a_0r^{\varepsilon}\right\}=\lim_{\varepsilon\to 0}\left\{\frac{d\left(a_0r^{\varepsilon}\right)}{d\varepsilon}\right\}=\lim_{\varepsilon\to 0}\left\{a_0r^{\varepsilon}\ln r\right\}=a_0\ln r$$

- Whenever a degeneracy occurs at a special value of *n*, the deficit is possibly made up by using additional terms obtained from differentiating the original form w.r.t. *n* before allowing it to take the special value.

General Solution of Biharmonic Equation

- General solution with degeneracy

$$\psi(r,\theta)=\sum_{n=0}^{\infty}\left(f_n(r)\cos n\theta+g_n(r)\sin n\theta\right)$$

$$=\left(a_0+a_1\ln r+a_2r^2+a_3r^2\ln r\right)+\left(b_0+b_1\ln r+b_2r^2+b_3r^2\ln r\right)\sin\left((0)(\theta)\right)$$

$$+\left(a_{11}r+a_{12}r\ln r+a_{13}/r+a_{14}r^3\right)\cos\theta+\left(b_{11}r+b_{12}r\ln r+b_{13}/r+b_{14}r^3\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left(a_{n1}r^n+a_{n2}r^{2+n}+a_{n3}/r^n+a_{n4}/r^{n-2}\right)\cos n\theta+\sum_{n=2}^{\infty}\left(b_{n1}r^n+b_{n2}r^{2+n}+b_{n3}/r^n+b_{n4}/r^{n-2}\right)\sin n\theta$$

- For other degenerate terms

$$\psi=\lim_{\varepsilon\to 0}\left\{\left(b_0+b_1\ln r+b_2r^2+b_3r^2\ln r\right)\sin\varepsilon\theta+a'_{11}r^{(1+\varepsilon)}\cos(1+\varepsilon)\theta+b'_{11}r^{(1+\varepsilon)}\sin(1+\varepsilon)\theta\right\}$$

$$=\lim_{\varepsilon\to 0}\left\{\begin{array}{l}\cdots+\left(b_0+b_1\ln r+b_2r^2+b_3r^2\ln r\right)\theta\cos\varepsilon\theta+a'_{11}r^{(1+\varepsilon)}\ln r\cos(1+\varepsilon)\theta\\ -a'_{11}r^{(1+\varepsilon)}\theta\sin(1+\varepsilon)\theta+b'_{11}r^{(1+\varepsilon)}\ln r\sin(1+\varepsilon)\theta+b'_{11}r^{(1+\varepsilon)}\theta\cos(1+\varepsilon)\theta\end{array}\right\}$$

$$=\left(b_0+b_1\ln r+b_2r^2+b_3r^2\ln r\right)\theta+a'_{11}r\ln r\cos\theta-a'_{11}r\theta\sin\theta+b'_{11}r\ln r\sin\theta+b'_{11}r\theta\cos\theta$$

- Red-colored terms produce multi-valued displacements.

General Solution of Biharmonic Equation

- The general Michell solution:

$$\psi(r,\theta) = a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r + \left(a_4 + a_5 \ln r + a_6 r^2 + a_7 r^2 \ln r\right)\theta$$
$$+\left(a_{12} r \ln r + a_{13}/r + a_{14} r^3 + a_{15} r\theta\right)\cos\theta + \left(b_{12} r \ln r + b_{13}/r + b_{14} r^3 + b_{15} r\theta\right)\sin\theta$$
$$+\sum_{n=2}^{\infty}\left(a_{n1} r^n + a_{n2} r^{2+n} + a_{n3}/r^n + a_{n4}/r^{n-2}\right)\cos n\theta + \sum_{n=2}^{\infty}\left(b_{n1} r^n + b_{n2} r^{2+n} + b_{n3}/r^n + b_{n4}/r^{n-2}\right)\sin n\theta$$

- Satisfaction of boundary conditions

$$\sigma_r(a,\theta) = F_1(\theta), \quad \tau_{r\theta}(a,\theta) = F_2(\theta), \quad \sigma_r(b,\theta) = F_3(\theta), \quad \tau_{r\theta}(b,\theta) = F_4(\theta)$$

- Expanding the functions as Fourier series in θ:

$$F_j(\theta) = \sum_{n=0}^{\infty} C_{nj} \cos(n\theta) + \sum_{n=1}^{\infty} D_{nj} \sin(n\theta), \qquad j = 1,2,3,4$$

General Solution of Biharmonic Equation

- Stress field

$$\sigma_r = \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = a_1/r^2 + 2a_2 + a_3\left(2\ln r + 1\right) + \left(b_1/r^2 + 2b_2 + b_3\left(2\ln r + 1\right)\right)\theta$$

$$+\left(a_{12}/r - 2a_{13}/r^3 + 2a_{14}r\right)\cos\theta - 2a_{15}\sin\theta/r \;\; + \left(b_{12}/r - 2b_{13}/r^3 + 2b_{14}r\right)\sin\theta + 2b_{15}\cos\theta/r$$

$$+\sum_{n=2}^{\infty}\left\{-a_{n1}n\left(n-1\right)r^{n-2} - a_{n2}\left(n+1\right)\left(n-2\right)r^n - a_{n3}n\left(n+1\right)/r^{n+2} - a_{n4}\left(n+2\right)\left(n-1\right)/r^n\right\}\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left\{-b_{n1}n\left(n-1\right)r^{n-2} - b_{n2}\left(n+1\right)\left(n-2\right)r^n - b_{n3}n\left(n+1\right)/r^{n+2} - b_{n4}\left(n+2\right)\left(n-1\right)/r^n\right\}\sin n\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = b_0/r^2 + b_1\left(\ln r - 1\right)/r^2 - b_2 - b_3\left(\ln r + 1\right)$$

$$+\left(a_{12}/r - 2a_{13}/r^3 + 2a_{14}r\right)\sin\theta + \left(-b_{12}/r + 2b_{13}/r^3 - 2b_{14}r\right)\cos\theta$$

$$+\sum_{n=2}^{\infty}\left\{a_{n1}n\left(n-1\right)r^{n-2} + a_{n2}n\left(n+1\right)r^n - a_{n3}n\left(n+1\right)/r^{n+2} - a_{n4}n\left(n-1\right)/r^n\right\}\sin n\theta$$

$$+\sum_{n=2}^{\infty}\left\{-b_{n1}n\left(n-1\right)r^{n-2} - b_{n2}n\left(n+1\right)r^n + b_{n3}n\left(n+1\right)/r^{n+2} + b_{n4}n\left(n-1\right)/r^n\right\}\cos n\theta$$

General Solution of Biharmonic Equation

- Stress field

$$\sigma_\theta = \frac{\partial^2 \psi}{\partial r^2}$$

$$= -a_1/r^2 + 2a_2 + a_3\left(2\ln r + 3\right) + \left(-b_1/r^2 + 2b_2 + b_3\left(2\ln r + 3\right)\right)\theta$$

$$+\left(a_{12}/r + 2a_{13}/r^3 + 6a_{14}r\right)\cos\theta + \left(b_{12}/r + 2b_{13}/r^3 + 6b_{14}r\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left\{\begin{array}{l} a_{n1}n\left(n-1\right)r^{n-2} + a_{n2}\left(n+2\right)\left(n+1\right)r^{n} \\ +a_{n3}n\left(n+1\right)/r^{n+2} + a_{n4}\left(n-1\right)\left(n-2\right)/r^{n} \end{array}\right\}\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left\{\begin{array}{l} b_{n1}n\left(n-1\right)r^{n-2} + b_{n2}\left(n+2\right)\left(n+1\right)r^{n} \\ +b_{n3}n\left(n+1\right)/r^{n+2} + b_{n4}\left(n-1\right)\left(n-2\right)/r^{n} \end{array}\right\}\sin n\theta$$

General Solution of Biharmonic Equation

- Displacement field

$$
\begin{aligned}
2Gu_r = &\left\{-a_1/r + a_2(\kappa-1)r + a_3\left((\kappa-1)\ln r - 1\right)r\right\} \\
&+\left\{-b_1/r + b_2(\kappa-1)r + b_3\left((\kappa-1)r\ln r - r\right)\right\}\theta \\
&+a_{12}\left((\kappa+1)\theta\sin\theta + (\kappa-1)\ln r\cos\theta - \cos\theta\right)/2 \\
&+a_{13}\cos\theta/r^2 + a_{14}(\kappa-2)r^2\cos\theta \\
&+a_{15}\left((\kappa-1)\theta\cos\theta - (\kappa+1)\ln r\sin\theta + \sin\theta\right)/2 \\
&+b_{12}\left(-(\kappa+1)\theta\cos\theta + (\kappa-1)\ln r\sin\theta - \sin\theta\right)/2 \\
&+b_{13}\sin\theta/r^2 + b_{14}(\kappa-2)r^2\sin\theta \\
&+b_{15}\left((\kappa-1)\theta\sin\theta + (\kappa+1)\ln r\cos\theta - \cos\theta\right)/2 \\
&+\sum_{n=2}^{\infty}\left\{-a_{n1}nr^{n-1} + a_{n2}(\kappa-n-1)r^{n+1} + a_{n3}nr^{-n-1} + a_{n4}(\kappa+n-1)r^{-n+1}\right\}\cos n\theta \\
&+\sum_{n=2}^{\infty}\left\{-b_{n1}nr^{n-1} + b_{n2}(\kappa-n-1)r^{n+1} + b_{n3}nr^{-n-1} + b_{n4}(\kappa+n-1)r^{-n+1}\right\}\sin n\theta
\end{aligned}
$$

- Red-colored terms correspond to multi-valued displacements.

General Solution of Biharmonic Equation

- Displacement field

$$2Gu_\theta = a_3(\kappa+1)r\theta - b_0/r - b_1(2\ln r+1)/(4r) - b_2(\kappa+1)r\ln r$$

$$+b_3\left((\kappa+1)r\theta^2/2 - \kappa r(\ln r)^2/2\right)$$

$$+a_{12}\left((\kappa+1)\theta\cos\theta - (\kappa-1)\ln r\sin\theta - \sin\theta\right)/2$$

$$+a_{13}\sin\theta/r^2 + a_{14}(\kappa+2)r^2\sin\theta$$

$$-a_{15}\left((\kappa-1)\theta\sin\theta + (\kappa+1)\ln r\cos\theta + \cos\theta\right)/2$$

$$+b_{12}\left((\kappa+1)\theta\sin\theta + (\kappa-1)\ln r\cos\theta + \cos\theta\right)/2$$

$$-b_{13}\cos\theta/r^2 - b_{14}(\kappa+2)r^2\cos\theta$$

$$+b_{15}\left((\kappa-1)\theta\cos\theta - (\kappa+1)\ln r\sin\theta - \sin\theta\right)/2$$

$$+\sum_{n=2}^{\infty}\left\{a_{n1}nr^{n-1} + a_{n2}(\kappa+n+1)r^{n+1} + a_{n3}nr^{-n-1} - a_{n4}(\kappa-n+1)r^{-n+1}\right\}\sin n\theta$$

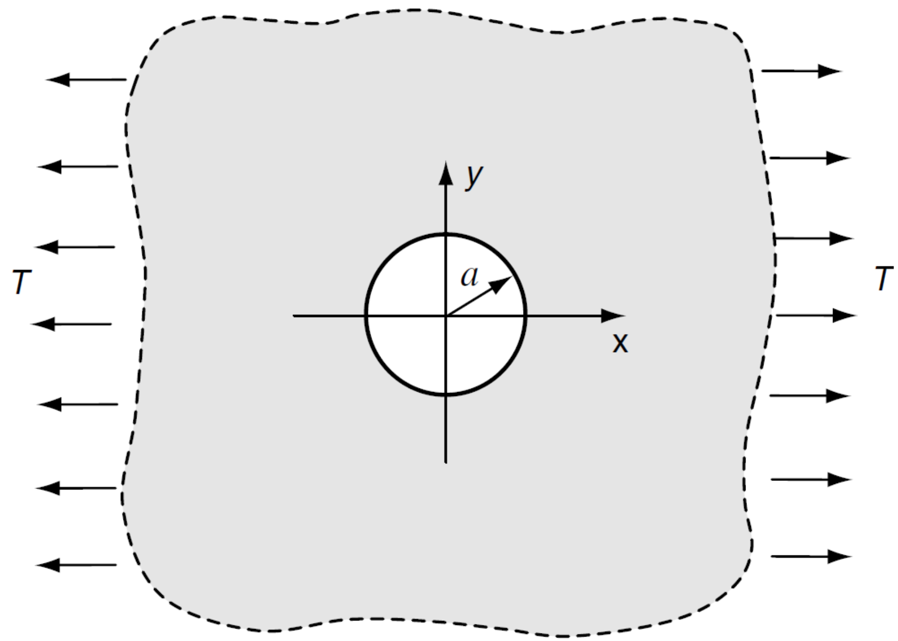
$$+\sum_{n=2}^{\infty}\left\{-b_{n1}nr^{n-1} - b_{n2}(\kappa+n+1)r^{n+1} - b_{n3}nr^{-n-1} + b_{n4}(\kappa-n+1)r^{-n+1}\right\}\cos n\theta$$

Hole in an Infinite Medium under Uniaxial Remote Tension

- A circular hole in an infinite plane under remote uniaxial loading
- BCs on hole surface

$$\sigma_r(a,\theta) = \tau_{r\theta}(a,\theta) = 0$$

- Remote BCs



$$\left[\sigma_{ij}(\infty)\right] = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left[\sigma'_{ij}(\infty)\right] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} T\cos^2\theta & -T\sin\theta\cos\theta \\ -T\sin\theta\cos\theta & T\sin^2\theta \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_r(\infty,\theta) &= \frac{T}{2}\left(1+\cos 2\theta\right) \\ \tau_{r\theta}(\infty,\theta) &= -\frac{T}{2}\sin 2\theta \\ \sigma_\theta(\infty,\theta) &= \frac{T}{2}\left(1-\cos 2\theta\right) \end{aligned}}$$

- **For remote field only**

$$\psi = \frac{1}{2}Ty^2 = \frac{1}{2}Tr^2\sin^2\theta = \frac{1}{4}Tr^2\left(1-\cos 2\theta\right)$$

- Due to the hole, additional (axisymmetric and cos2θ) terms are needed.
- The resultant stresses from additional terms must decay to zero as r tends to infinity.

Hole in an Infinite Medium under Uniaxial Remote Tension

- **A trial solution** that includes the axisymmetric and $\cos 2\theta$ terms from the general Michell solution

$$\psi = a_0 + a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r + \left(b_0 + b_1 \ln r + b_2 r^2 + b_3 r^2 \ln r\right)\theta$$

$$+\begin{pmatrix} a_{12} r \ln r + \dfrac{a_{13}}{r} \\ +a_{14} r^3 + a_{15} r\theta \end{pmatrix}\cos\theta + \begin{pmatrix} b_{12} r \ln r + \dfrac{b_{13}}{r} \\ +b_{14} r^3 + b_{15} r\theta \end{pmatrix}\sin\theta$$

$$+\sum_{n=2}^{\infty}\begin{pmatrix} a_{n1} r^n + a_{n2} r^{2+n} \\ +a_{n3} r^{-n} + a_{n4} r^{2-n} \end{pmatrix}\cos n\theta + \sum_{n=2}^{\infty}\begin{pmatrix} b_{n1} r^n + b_{n2} r^{2+n} \\ +b_{n3} r^{-n} + b_{n4} r^{2-n} \end{pmatrix}\sin n\theta.$$

$$\sigma_r = \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \psi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right).$$

$$\psi = a_0 + a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r + \left(a_{21} r^2 + a_{22} r^4 + a_{23} r^{-2} + a_{24}\right)\cos 2\theta.$$

Hole in an Infinite Medium under Uniaxial Remote Tension

- Stress field

$$\begin{cases} \sigma_r = \cancel{a_3(1+2\ln r)} + 2a_2 + \dfrac{a_1}{r^2} - \left(2a_{21} + \dfrac{6a_{23}}{r^4} + \dfrac{4a_{24}}{r^2}\right)\cos 2\theta \\ \sigma_\theta = \cancel{a_3(3+2\ln r)} + 2a_2 - \dfrac{a_1}{r^2} + \left(2a_{21} \cancel{+12a_{22}r^4} + \dfrac{6a_{23}}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} = \left(2a_{21} \cancel{+6a_{22}r^2} - \dfrac{6a_{23}}{r^4} - \dfrac{2a_{24}}{r^2}\right)\sin 2\theta \end{cases}$$

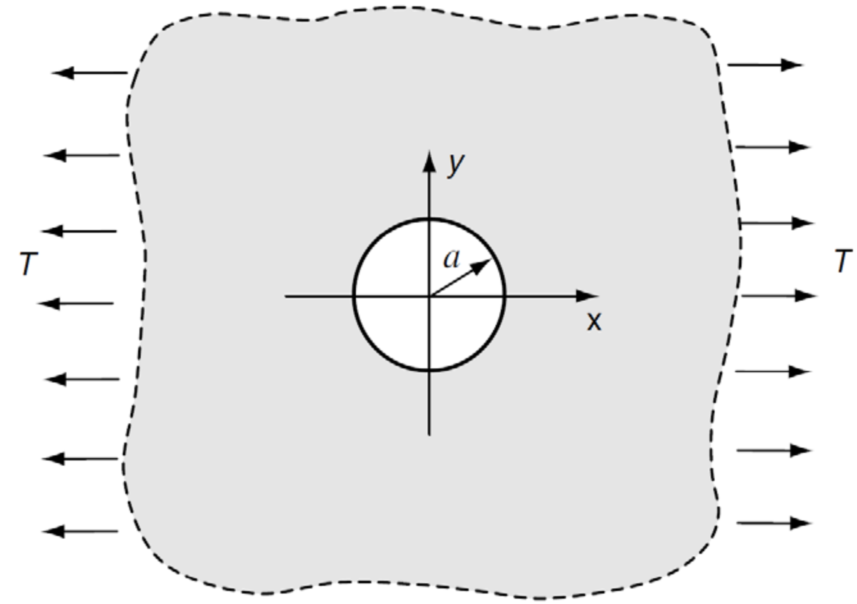
- For finite stress at infinity: $a_3 = a_{22} = 0.$
- Applying the BCs

$$\begin{cases} \sigma_r(a,\theta) = 0 \;\Rightarrow\; 2a_2 + \dfrac{a_1}{a^2} = 0,\; 2a_{21} + \dfrac{6a_{23}}{a^4} + \dfrac{4a_{24}}{a^2} = 0 \\ \tau_{r\theta}(a,\theta) = 0 \;\Rightarrow\; 2a_{21} - \dfrac{6a_{23}}{a^4} - \dfrac{2a_{24}}{a^2} = 0 \\ \sigma_r(\infty,\theta) = \dfrac{T}{2}(1+\cos 2\theta) \;\Rightarrow\; 2a_2 = \dfrac{T}{2}, -2a_{21} = \dfrac{T}{2} \\ \tau_{r\theta}(\infty,\theta) = -\dfrac{T}{2}\sin 2\theta \;\Rightarrow\; 2a_{21} = -\dfrac{T}{2} \end{cases} \Rightarrow \begin{cases} a_1 = -\dfrac{T}{2}a^2 \\ a_2 = \dfrac{T}{4} \\ a_{21} = -\dfrac{T}{4} \\ a_{23} = -\dfrac{T}{4}a^2 \\ a_{24} = \dfrac{T}{2}a^2 \end{cases}$$

Hole in an Infinite Medium under Uniaxial Remote Tension

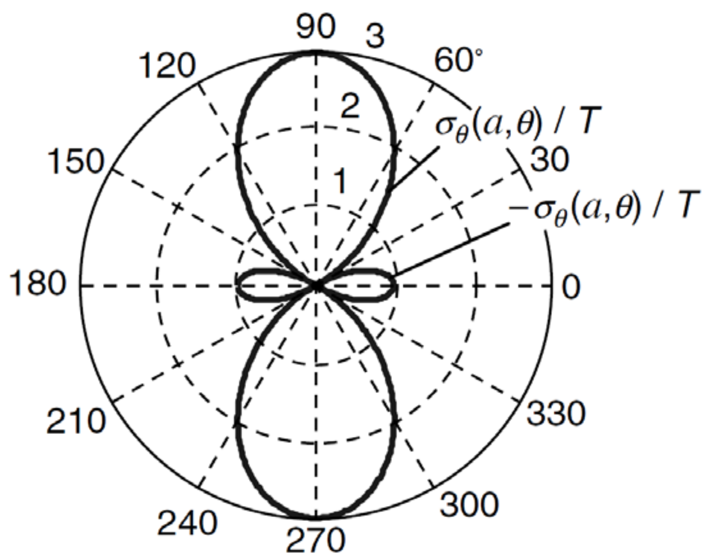
- Stress field

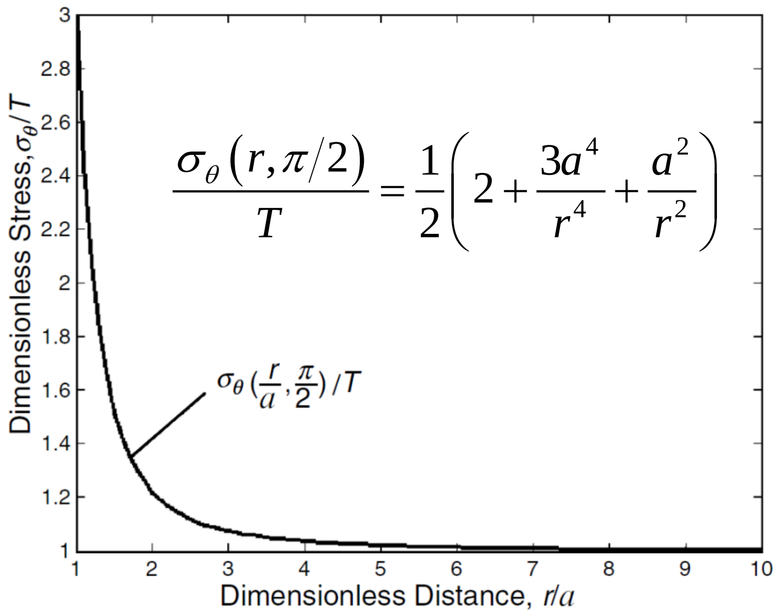
$$\begin{cases} \sigma_r = \dfrac{T}{2}\left(1-\dfrac{a^2}{r^2}\right)+\dfrac{T}{2}\left(1+\dfrac{3a^4}{r^4}-\dfrac{4a^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta = \dfrac{T}{2}\left(1+\dfrac{a^2}{r^2}\right)-\dfrac{T}{2}\left(1+\dfrac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} = -\dfrac{T}{2}\left(1-\dfrac{3a^4}{r^4}+\dfrac{2a^2}{r^2}\right)\sin 2\theta \end{cases}$$



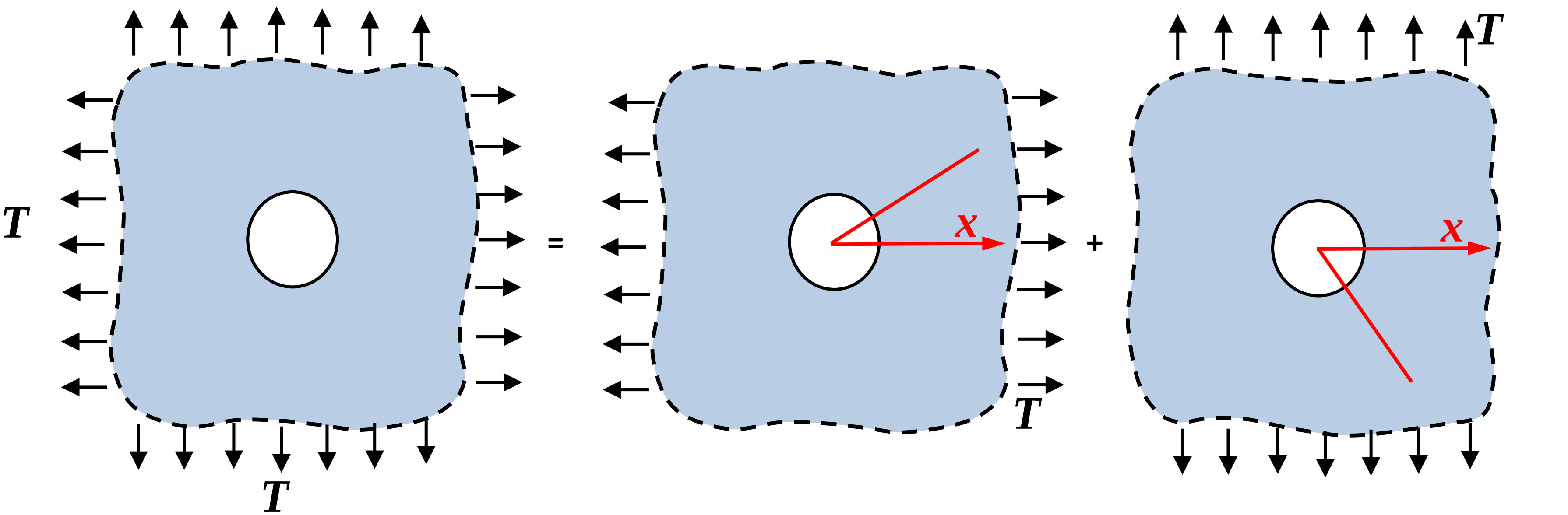
$$\sigma_\theta(a,\theta) = T\left(1-2\cos 2\theta\right)$$

$$\Rightarrow \sigma_\theta(a,0) = -T\,,\ \sigma_\theta(a,30^o) = 0,\ \ \sigma_{\max} = \sigma_\theta\left(a,\pm\pi/2\right) = 3T$$





Equal Biaxial Loading by Superposition



Equal Biaxial Tension Case

$$\sigma_r = T\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = T\left(1 + \frac{a_1^2}{r^2}\right)$$

$$\sigma_{\max} = (\sigma_\theta)_{\max} = \sigma_\theta(r_1) = 2T$$

Uniaxial Tension Case

$$\sigma_r = \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{T}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{T}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a, \pm\pi/2) = 3T$$

$$(\sigma_\theta)_{\min} = \sigma_\theta(a, 0) = \sigma_\theta(a, \pi) = -T$$

Uniaxial Tension Case

$$\theta \to \theta + \pi/2 \Rightarrow \sin\theta \to \cos\theta, \cos\theta \to -\sin\theta$$

$$\Rightarrow \cos 2\theta \to -\cos 2\theta, \sin 2\theta \to -\sin 2\theta$$

$$\sigma_r = \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right) - \frac{T}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

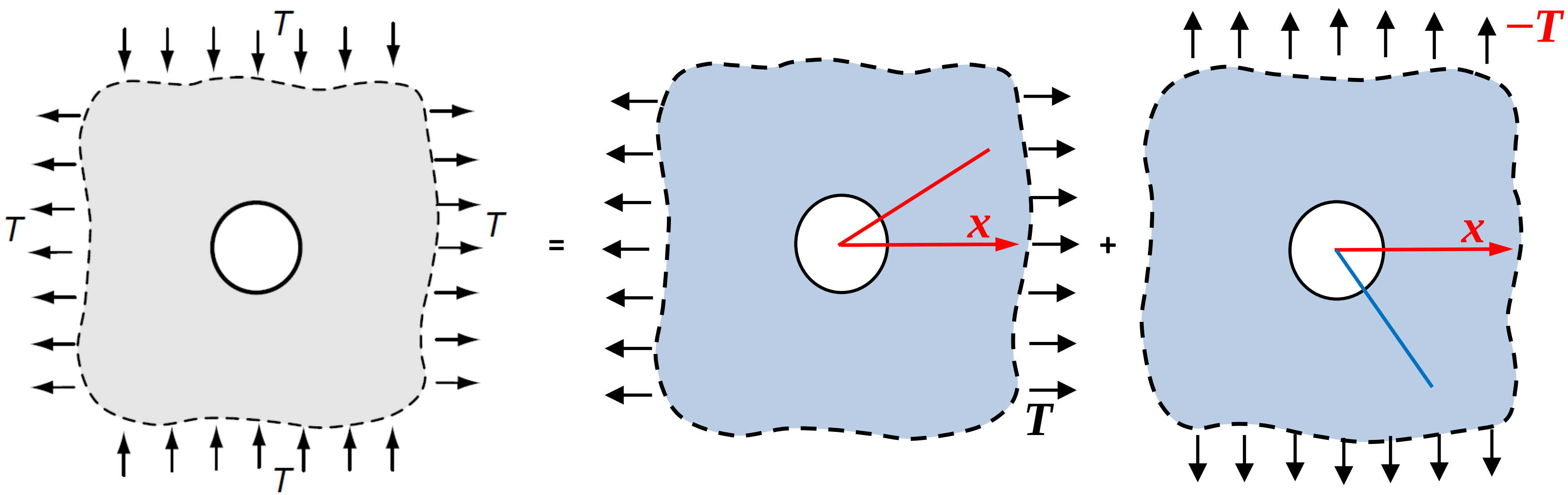
$$\sigma_\theta = \frac{T}{2}\left(1 + \frac{a^2}{r^2}\right) + \frac{T}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = +\frac{T}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a, 0) = \sigma_\theta(a, \pi) = 3T$$

$$(\sigma_\theta)_{\min} = \sigma_\theta(a, \pm\pi/2) = -T$$

Opposite Biaxial Loading by Superposition



Opposite Biaxial Tension | **Uniaxial Tension** | **Uniaxial Compression**

Opposite Biaxial Tension

$$\sigma_r = T\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = -T\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -T\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

$$(\sigma_\theta)_{\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -4T$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a,\pm\pi/2) = 4T$$

Uniaxial Tension

$$\sigma_r = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a,\pm\pi/2) = 3T$$

$$(\sigma_\theta)_{\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -T$$

Uniaxial Compression

$$\sigma_r = \frac{-T}{2}\left(1-\frac{a^2}{r^2}\right)-\frac{-T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

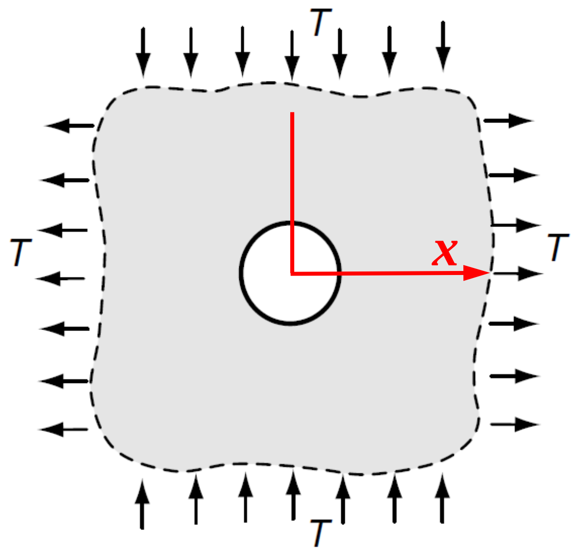
$$\sigma_\theta = \frac{-T}{2}\left(1+\frac{a^2}{r^2}\right)+\frac{-T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = +\frac{-T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

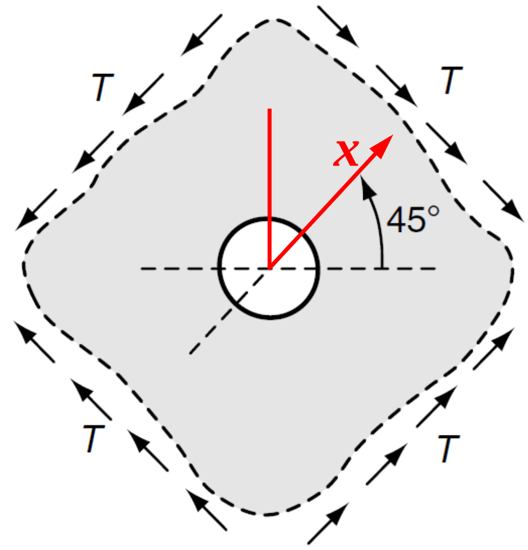
$$(\sigma_\theta)_{\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -3T$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a,\pm\pi/2) = T$$

Opposite Biaxial Loading / Shear Loading



(a) Biaxial Loading



(b) Shear Loading

$$\sigma_r = T\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = -T\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -T\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

$$\left(\sigma_\theta\right)_{\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -4T$$

$$\left(\sigma_\theta\right)_{\max} = \sigma_\theta\left(a,\pm\pi/2\right) = 4T$$

$$\theta \to \theta+\pi/4 \Rightarrow \sin 2\theta \to \cos 2\theta, \cos 2\theta \to -\sin 2\theta$$

$$\sigma_r = -T\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\sin 2\theta$$

$$\sigma_\theta = +T\left(1+\frac{3a^4}{r^4}\right)\sin 2\theta$$

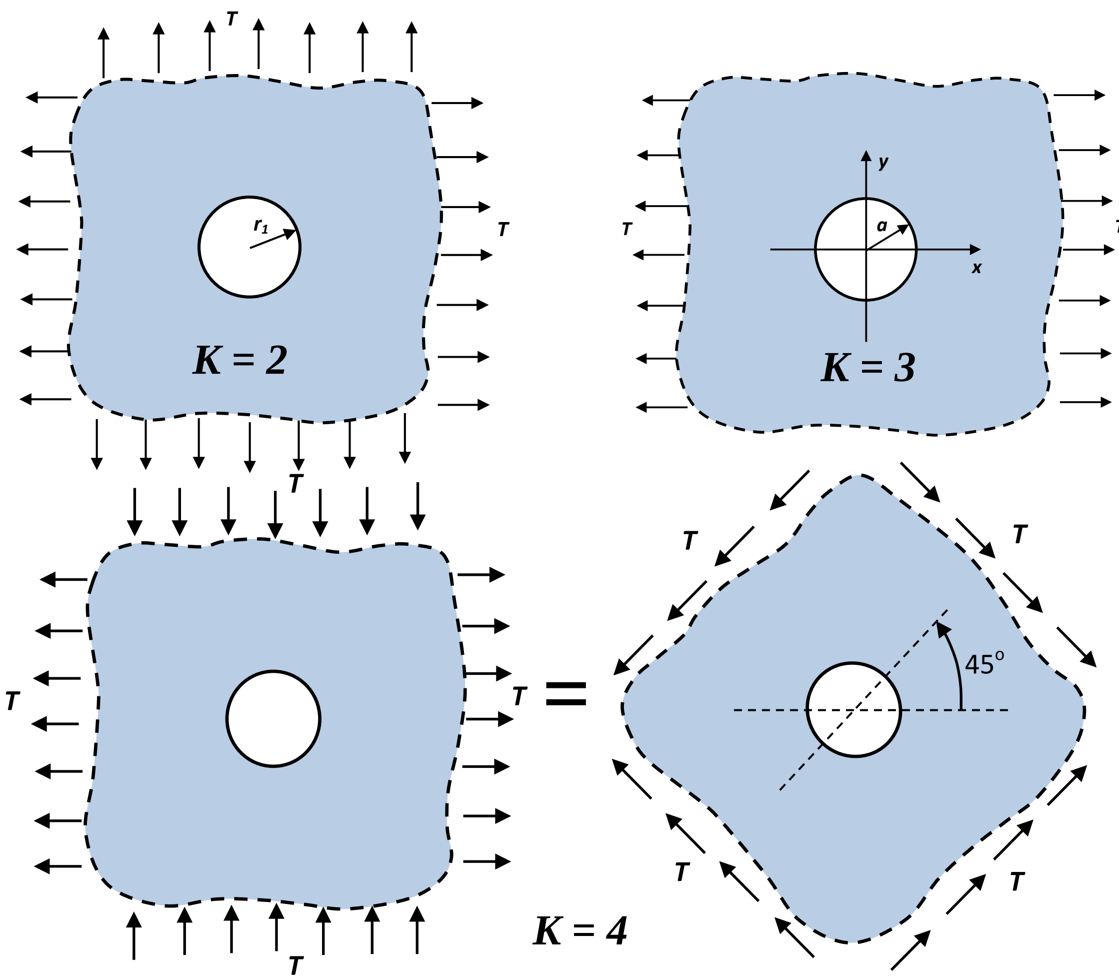
$$\tau_{r\theta} = -T\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\cos 2\theta$$

$$\left(\sigma_\theta\right)_{\min} = \sigma_\theta(a,-\pi/4) = -4T$$

$$\left(\sigma_\theta\right)_{\max} = \sigma_\theta\left(a,\pi/4\right) = 4T$$

Stress Concentration around a Hole

- The stress concentration can be measured by the stress concentration coefficients that are the ratios between the most severe stress at the critical point (or termed hot spot) and the remote stress.



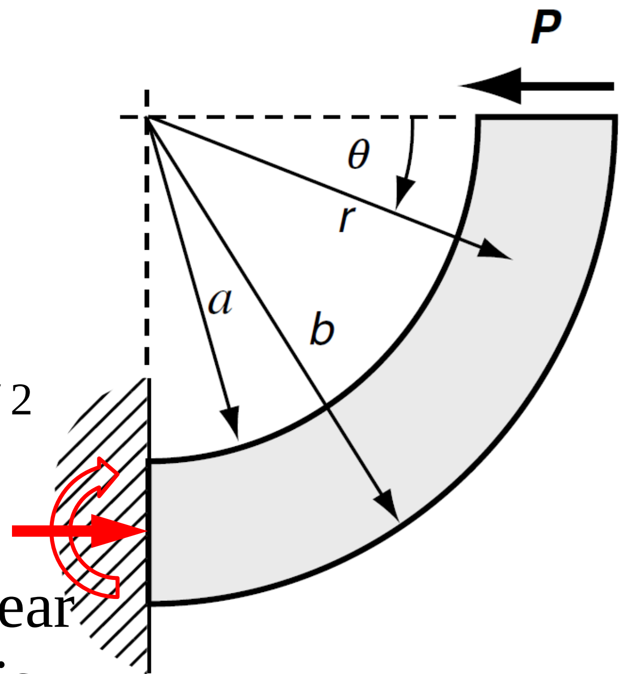
$$K = \frac{\sigma_{\max}}{\sigma}$$

Curved Cantilever Beams with End Loads

- Boundary conditions

$$\begin{cases}\sigma_r(a,\theta)=\sigma_r(b,\theta)=0\\ \tau_{r\theta}(a,\theta)=\tau_{r\theta}(b,\theta)=0\end{cases}$$

$$\begin{cases}\int_a^b \tau_{r\theta}(r,0)dr=P\\ \int_a^b \sigma_\theta(r,0)dr=\int_a^b \sigma_\theta(r,0)rdr=0\end{cases}\qquad \begin{cases}\int_a^b \sigma_\theta(r,\pi/2)dr=-P\\ \int_a^b \sigma_\theta(r,\pi/2)rdr=P(a+b)/2\\ \int_a^b \tau_{r\theta}(r,\pi/2)dr=0\end{cases}$$



- Static equilibrium suggests: the internal shear force varies with $P\cos\theta$ along the beam axis.

$$\tau_{r\theta}=b_0\frac{1}{r^2}+b_1\frac{1}{r^2}(\ln r-1)-b_2-b_3(\ln r+1)$$
$$+\left(a_{12}\frac{1}{r}-2a_{13}\frac{1}{r^3}+2a_{14}r\right)\sin\theta+\left(-b_{12}\frac{1}{r}+2b_{13}\frac{1}{r^3}-2b_{14}r\right)\cos\theta$$
$$+\sum_{n=2}^{\infty}\left(a_{n1}n(n-1)r^{n-2}+a_{n2}n(n+1)r^n-a_{n3}n(n+1)r^{-n-2}-a_{n4}n(n-1)r^{-n}\right)\sin n\theta$$
$$+\sum_{n=2}^{\infty}\left(-b_{n1}n(n-1)r^{n-2}-b_{n2}n(n+1)r^n+b_{n3}n(n+1)r^{-n-2}+b_{n4}n(n-1)r^{-n}\right)\cos n\theta$$

Curved Cantilever Beams with End Loads

- Corresponding Airy Stress function: $\psi = \left(b_{12} r \ln r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta \right) \sin\theta$
- Applying the BCs

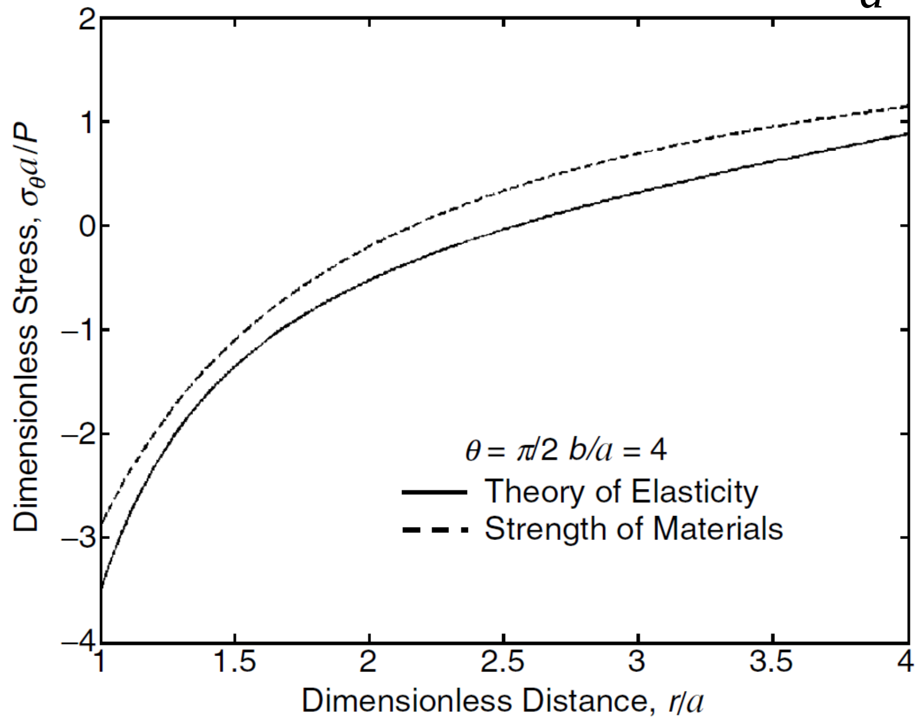
$$b_{12} = -\frac{P}{N}\left(a^2 + b^2\right),\ b_{13} = -\frac{Pa^2b^2}{2N},\ b_{14} = \frac{P}{2N},\ b_{15} = 0,\ \text{where}\ N = a^2 - b^2 + \left(a^2 + b^2\right)\ln\frac{b}{a}$$

- Stress field

$$\sigma_r = \frac{P}{N}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2 + b^2}{r} \right)\sin\theta$$

$$\sigma_\theta = \frac{P}{N}\left(3r - \frac{a^2b^2}{r^3} - \frac{a^2 + b^2}{r} \right)\sin\theta$$

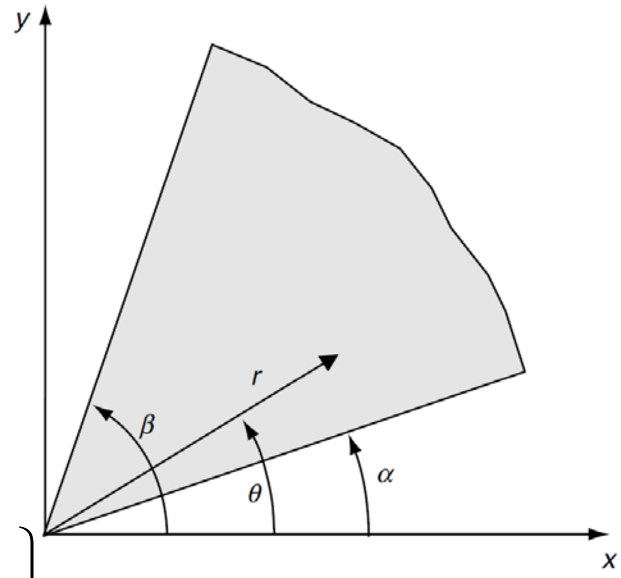
$$\tau_{r\theta} = -\frac{P}{N}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2 + b^2}{r} \right)\cos\theta$$



- The corresponding solution for an axial force applied at the end $\theta = 0$ is obtained through superposing the present solution (interchanging sine and cosine) with the pure bending solution.

Wedge Problem: Power Law Tractions

- We first consider the case in which the tractions on the boundaries of a wedge vary with r^n:



- Since $\sigma_\theta = \dfrac{\partial^2 \psi}{\partial r^2}$

$$\psi = \sum_{n=2}^{\infty}\left(a_{n1}r^n + a_{n2}r^{2+n} + a_{n3}r^{-n} + a_{n4}r^{2-n}\right)\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left(b_{n1}r^n + b_{n2}r^{2+n} + b_{n3}r^{-n} + b_{n4}r^{2-n}\right)\sin n\theta$$

- Hence

$$\psi = r^{n+2}\begin{Bmatrix} a_{n2}\cos n\theta + b_{n2}\sin n\theta \\ +a_{(n+2)1}\cos\left(n+2\right)\theta + b_{(n+2)1}\sin\left(n+2\right)\theta \end{Bmatrix}, \quad n \neq 0, -2$$

- For example, the second term in the above degenerates when n = 0. A special solution is obtained by differentiating with respect to n, before enforcing to the limit $n \to 0$.

$$\psi = \lim_{n\to 0}\frac{d\left(b_{n2}r^{n+2}\sin n\theta\right)}{dn} = \lim_{n\to 0}\left(b_{n2}r^{n+2}\theta\cos n\theta\right) = b_2 r^2\theta$$

Wedge Problem: Uniform Boundary Loading

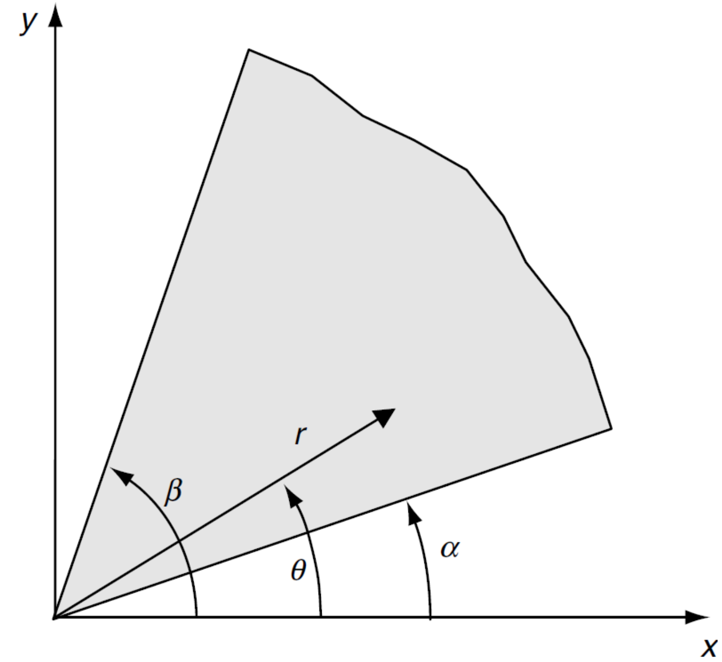
- Use general stress function solution to include terms **that give uniform stresses on the boundaries**

$$\sigma_r = 2a_2 + 2b_2\theta - 2a_{21}\cos 2\theta - 2b_{21}\sin 2\theta$$

$$\sigma_\theta = 2a_2 + 2b_2\theta + 2a_{21}\cos 2\theta + 2b_{21}\sin 2\theta$$

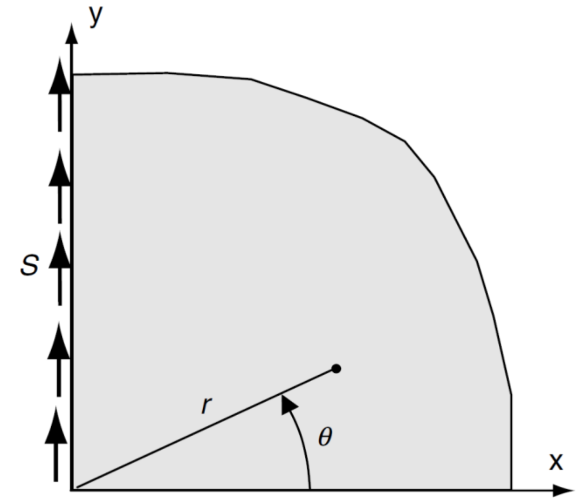
$$\tau_{r\theta} = -b_2 + 2a_{21}\sin 2\theta - 2b_{21}\cos 2\theta$$

$$\psi = r^2\left(a_2 + b_2\theta + a_{21}\cos 2\theta + b_{21}\sin 2\theta\right)$$



- In general, the above equations permit us to solve the problem of the wedge with any combination of four independent traction components on the faces.

Quarter Plane Example



- BCs

$$\theta = 0: \ \sigma_\theta(r,0) = \tau_{r\theta}(r,0) = 0$$

$$\theta = \pi/2: \ \sigma_\theta(r,\pi/2) = 0, \ \tau_{r\theta}(r,\pi/2) = S$$

- Applying the BCs

$$\begin{cases} 0 = \sigma_\theta(r,0) = 2a_2 + 2a_{21} \\ 0 = \tau_{r\theta}(r,0) = -b_2 - 2b_{21} \\ 0 = \sigma_\theta(r,\pi/2) = 2a_2 + \pi b_2 - 2a_{21} \\ S = \tau_{r\theta}(r,\pi/2) = -b_2 + 2b_{21} \end{cases} \Rightarrow \begin{cases} a_2 = \dfrac{S\pi}{8}, b_2 = -\dfrac{S}{2} \\ a_{21} = -\dfrac{S\pi}{8}, b_{21} = \dfrac{S}{4} \end{cases}$$

- Stress field

$$\sigma_r = \frac{S}{2}\left(\frac{\pi}{2} - 2\theta + \frac{\pi}{2}\cos 2\theta - \sin 2\theta\right), \quad \sigma_\theta = \frac{S}{2}\left(\frac{\pi}{2} - 2\theta - \frac{\pi}{2}\cos 2\theta + \sin 2\theta\right)$$

$$\tau_{r\theta} = \frac{S}{2}\left(1 - \cos 2\theta - \frac{\pi}{2}\sin 2\theta\right), \quad \boxed{\psi = \frac{S}{2}r^2\left(\frac{\pi}{4} - \theta - \frac{\pi}{4}\cos 2\theta + \frac{1}{2}\sin 2\theta\right)}$$

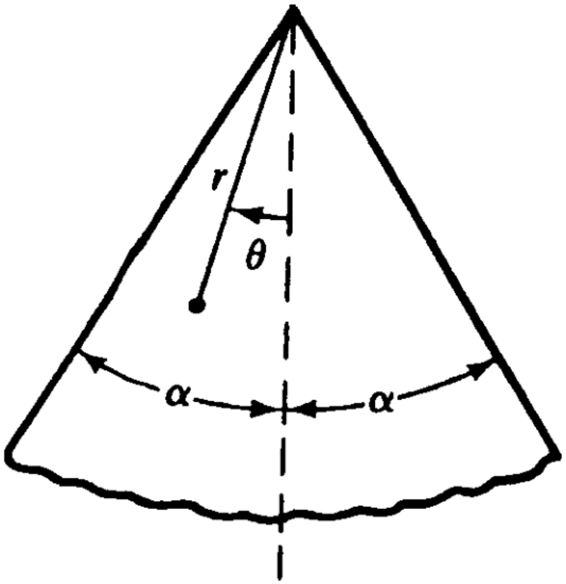
- Note the apparent inconsistency in the shear stress at the origin.

Wedge Problem: More General Uniform Loading

- Consider the most general uniform boundary loading

$$\sigma_\theta(r,\alpha) = N_1, \quad \tau_{r\theta}(r,\alpha) = S_1; \qquad \sigma_\theta(r,-\alpha) = N_2, \quad \tau_{r\theta}(r,-\alpha) = S_2$$

- Applying the BCs



$$\begin{cases} \sigma_\theta(r,\alpha) = 2a_2 + 2b_2\alpha + 2a_{21}\cos 2\alpha + 2b_{21}\sin 2\alpha = N_1 \\ \sigma_\theta(r,-\alpha) = 2a_2 - 2b_2\alpha + 2a_{21}\cos 2\alpha - 2b_{21}\sin 2\alpha = N_2 \\ \tau_{r\theta}(r,\alpha) = -b_2 + 2a_{21}\sin 2\alpha - 2b_{21}\cos 2\alpha = S_1 \\ \tau_{r\theta}(r,-\alpha) = -b_2 - 2a_{21}\sin 2\alpha - 2b_{21}\cos 2\alpha = S_2 \end{cases}$$

$$\Rightarrow \begin{cases} \boxed{\begin{array}{c} 4a_2 + 4a_{21}\cos 2\alpha = N_1 + N_2 \\ 4a_{21}\sin 2\alpha = S_1 - S_2 \end{array}}, \quad \text{symmetric solution} \\ \boxed{\begin{array}{c} 4b_2\alpha + 4b_{21}\sin 2\alpha = N_1 - N_2 \\ -2b_2 - 4b_{21}\cos 2\alpha = S_1 + S_2 \end{array}}, \quad \text{anti-symmetric solution} \end{cases}$$

- The solution of these equations is routine, but we note that there are two eigenvalues at which the matrix of coefficients is singular.

$$16\sin 2\alpha = 0 \quad \Rightarrow \quad \sin 2\alpha = 0 \quad \Rightarrow \quad 2\alpha = \pi, 2\pi$$

$$-16\alpha\cos 2\alpha + 8\sin 2\alpha = 0 \quad \Rightarrow \quad \tan 2\alpha = 2\alpha \quad \Rightarrow \quad 2\alpha = 1.43\pi = 257.4^0$$

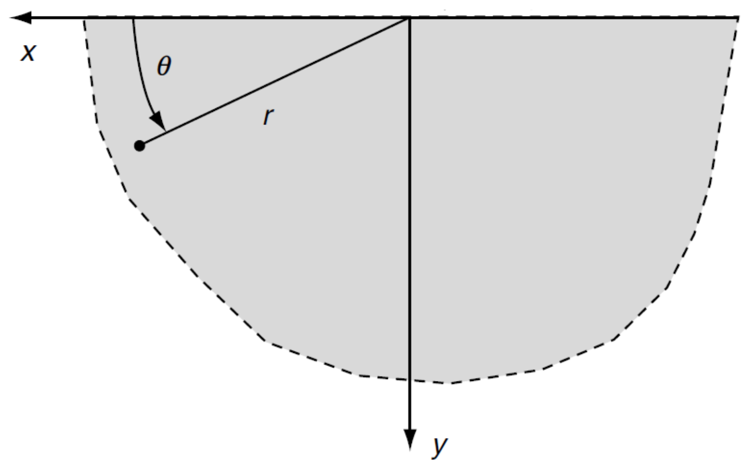
Half-Plane: Uniform Boundary Loading

- For the special case of a semi-infinite plane, i.e. $2\alpha = \pi$, the matrix of coefficients for the symmetric solution is singular.
- Additional terms must be developed to make up the deficit.

$$\psi = \lim_{n\to 0}\frac{d}{dn}\left\{a'_{n2}r^{n+2}\cos n\theta + a'_{(n+2)1}r^{n+2}\cos(n+2)\theta\right\}$$

$$= \lim_{n\to 0}\left\{\begin{aligned} & a'_{n2}r^{n+2}\ln r\cos n\theta - a'_{n2}r^{n+2}\theta\sin n\theta \\ & +a'_{(n+2)1}r^{n+2}\ln r\cos(n+2)\theta \\ & -a'_{(n+2)1}r^{n+2}\theta\sin(n+2)\theta \end{aligned}\right\}$$

$$= r^2\left\{a'_2\ln r + a'_{21}\left(\ln r\cos 2\theta - \theta\sin 2\theta\right)\right\}$$



$$\psi = r^2\left\{a_2 + b_2\theta + a_{21}\cos 2\theta + b_{21}\sin 2\theta + a'_2\ln r + a'_{21}\left(\ln r\cos 2\theta - \theta\sin 2\theta\right)\right\}$$

$$\Rightarrow \begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2} = 2a_2 + 2b_2\theta - 2a_{21}\cos 2\theta - 2b_{21}\sin 2\theta + a'_2(1+2\ln r) + a'_{21}(-2\ln r\cos 2\theta - 3\cos 2\theta + 2\theta\sin 2\theta), \\ \sigma_\theta = \dfrac{\partial^2\psi}{\partial r^2} = 2a_2 + 2b_2\theta + 2a_{21}\cos 2\theta + 2b_{21}\sin 2\theta + a'_2(3+2\ln r) + a'_{21}(2\ln r\cos 2\theta + 3\cos 2\theta - 2\theta\sin 2\theta), \\ \tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\right) = -b_2 + 2a_{21}\sin 2\theta - 2b_{21}\cos 2\theta + a'_{21}(2\ln r\sin 2\theta + 3\sin 2\theta + 2\theta\cos 2\theta). \end{cases}$$

Half-Plane: Uniform Shear over Half-Boundary

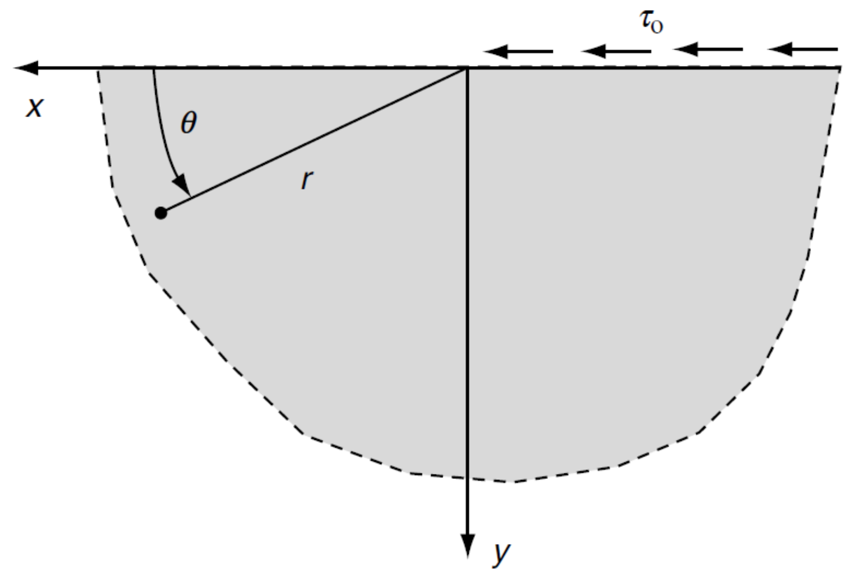
$$\boxed{\begin{array}{ll}\tau_{r\theta}(r,0)=0, & \tau_{r\theta}(r,\pi)=-\tau_0,\\ \sigma_\theta(r,0)=0, & \sigma_\theta(r,\pi)=0\end{array}}$$

$$\Rightarrow\begin{cases}-b_2-2b_{21}=0\\ -b_2-2b_{21}+2\pi a'_{21}=-\tau_0\\ 2a_2+2a_{21}+a'_2\left(3+2\ln r\right)+a'_{21}\left(2\ln r+3\right)=0\\ 2a_2+2\pi b_2+2a_{21}+a'_2\left(3+2\ln r\right)+a'_{21}\left(2\ln r+3\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}-b_2-2b_{21}=0\\ -b_2-2b_{21}+2\pi a'_{21}=-\tau_0\\ 2a_2+2a_{21}+3a'_2+3a'_{21}=0\\ 2a_2+2\pi b_2+2a_{21}+3a'_2+3a'_{21}=0\end{cases},\quad\begin{cases}\\ \\ a'_2+a'_{21}=0\\ a'_2+a'_{21}=0\end{cases}$$

$$\Rightarrow a'_{21}=-\frac{\tau_0}{2\pi},\ a'_2=\frac{\tau_0}{2\pi},\ b_2=0,\ b_{21}=0,\ a_{21}=-a_2$$

$$\Rightarrow\begin{cases}\psi=a_2r^2\left(1-\cos 2\theta\right)+\dfrac{\tau_0r^2}{2\pi}\left(\ln r\left(1-\cos 2\theta\right)+\theta\sin 2\theta\right)\\ \sigma_r=2a_2\left(1+\cos 2\theta\right)+\dfrac{\tau_0}{2\pi}\left(1+2\ln r\left(1+\cos 2\theta\right)+3\cos 2\theta-2\theta\sin 2\theta\right),\\ \sigma_\theta=2a_2\left(1-\cos 2\theta\right)+\dfrac{\tau_0}{2\pi}\left(3+2\ln r\left(1-\cos 2\theta\right)-3\cos 2\theta+2\theta\sin 2\theta\right),\\ \tau_{r\theta}=-2a_2\sin 2\theta-\dfrac{\tau_0}{2\pi}\left(2\ln r\sin 2\theta+3\sin 2\theta+2\theta\cos 2\theta\right)\end{cases}$$



- The parameter a_2 can take an arbitrary value without affecting the BCs at the plane boundary and might be determined from the far-field stress state.

Half-Plane: Uniform Pressure over Half-Boundary

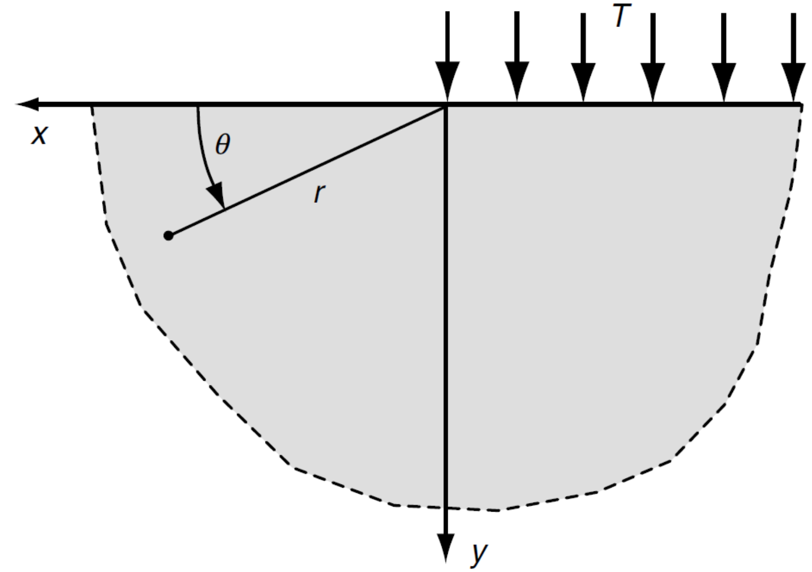
$$\boxed{\begin{aligned}&\tau_{r\theta}(r,0)=0, \quad \tau_{r\theta}(r,\pi)=0,\\ &\sigma_\theta(r,0)=0, \quad \sigma_\theta(r,\pi)=-T\end{aligned}}$$

$$\Rightarrow \begin{cases} -b_2-2b_{21}=0 \\ -b_2-2b_{21}+2\pi a'_{21}=0 \\ 2a_2+2a_{21}+a'_2\left(3+2\ln r\right)+a'_{21}\left(2\ln r+3\right)=0 \\ 2a_2+2\pi b_2+2a_{21}+a'_2\left(3+2\ln r\right)+a'_{21}\left(2\ln r+3\right)=-T \end{cases}$$

$$\Rightarrow \begin{cases} -b_2-2b_{21}=0 \\ -b_2-2b_{21}+2\pi a'_{21}=0 \\ 2a_2+2a_{21}+3a'_2+3a'_{21}=0 \\ 2a_2+2\pi b_2+2a_{21}+3a'_2+3a'_{21}=-T \end{cases}, \quad \begin{cases} a'_2+a'_{21}=0 \\ a'_2+a'_{21}=0 \end{cases}$$

$$\Rightarrow a'_{21}=0, a'_2=0, b_2=-\frac{T}{2\pi}, b_{21}=\frac{T}{4\pi}, a_{21}=-a_2$$

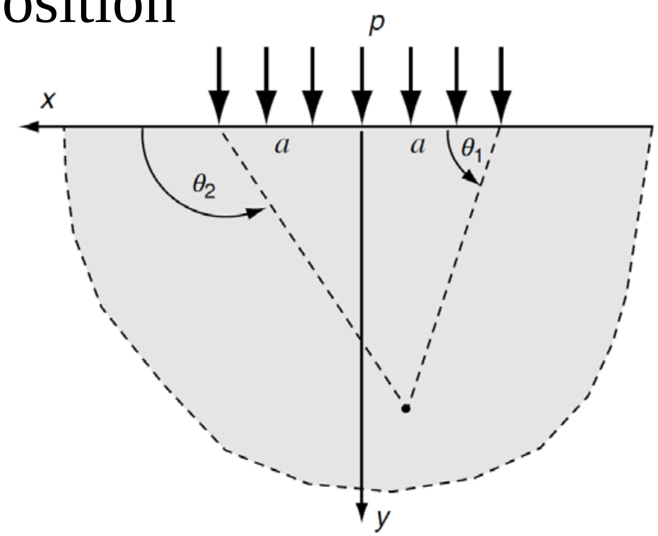
$$\begin{cases} \psi=a_2r^2\left(1-\cos 2\theta\right)-\dfrac{T}{4\pi}r^2\left(2\theta-\sin 2\theta\right) \\ \sigma_r=2a_2\left(1+\cos 2\theta\right)-\dfrac{T}{2\pi}\left(2\theta+\sin 2\theta\right), \\ \sigma_\theta=2a_2\left(1-\cos 2\theta\right)-\dfrac{T}{2\pi}\left(2\theta-\sin 2\theta\right), \\ \tau_{r\theta}=-2a_2\sin 2\theta+\dfrac{T}{2\pi}\left(1-\cos 2\theta\right) \end{cases}$$



- The parameter a_2 can take an arbitrary value without affecting the BCs at the plane boundary and might be determined from the far-field stress state.

Half-Plane: Pressure over Finite Boundary

- Construct Airy Stress Function by superposition



$$\psi = a_2\left(r_2^2 - r_1^2\right) - a_2\left(r_2^2\cos 2\theta_2 - r_1^2\cos 2\theta_1\right)$$
$$-\frac{p}{2\pi}\left(r_2^2\theta_2 - r_1^2\theta_1\right) + \frac{p}{4\pi}\left(r_2^2\sin 2\theta_2 - r_1^2\sin 2\theta_1\right)$$

- Three terms produce no stress (trivial)

$$r_2^2 - r_1^2 = \left[(x-a)^2 + y^2\right] - \left[(x+a)^2 + y^2\right] = -4a \qquad x$$

$$r_2^2\cos 2\theta_2 - r_1^2\cos 2\theta_1 = r_2^2\left(\cos^2\theta_2 - \sin^2\theta_2\right) - r_1^2\left(\cos^2\theta_1 - \sin^2\theta_1\right) = \left[(x-a)^2 - y^2\right] - \left[(x+a)^2 - y^2\right] = -4a$$

$$r_2^2\sin 2\theta_2 - r_1^2\sin 2\theta_1 = 2r_2^2\sin\theta_2\cos\theta_2 - 2r_1^2\sin\theta_1\cos\theta_1 = 2y(x-a) - 2y(x+a) = -4ay$$

- Only one term is meaningful: $\boxed{\psi = -\frac{p}{2\pi}\left(r_2^2\theta_2 - r_1^2\theta_1\right)}$

- Recall that

$$\text{For } \psi = -\frac{p}{2\pi}r^2\theta:\ \sigma_r = -\frac{p}{\pi}\theta,\ \sigma_\theta = -\frac{p}{\pi}\theta,\ \tau_{r\theta} = \frac{p}{2\pi}$$

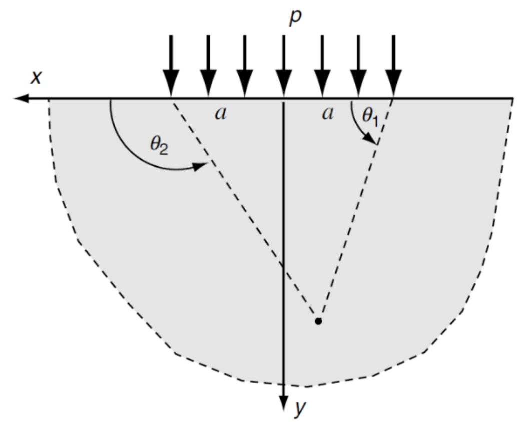
Half-Plane: Pressure over Finite Boundary

- Transforming from Polar Coordinates to RCC

$$\sigma_{ij} = Q_{ki} Q_{lj} \sigma'_{kl} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\sigma}'\mathbf{Q}^T$$

$$\Rightarrow \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_r & \tau_{r\theta} \\ \tau_{r\theta} & \sigma_\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\sigma_r - \sin\theta\tau_{r\theta} & \cos\theta\tau_{r\theta} - \sin\theta\sigma_\theta \\ \sin\theta\sigma_r + \cos\theta\tau_{r\theta} & \sin\theta\tau_{r\theta} + \cos\theta\sigma_\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\sigma_r + \sin^2\theta\sigma_\theta - \sin 2\theta\tau_{r\theta} & \frac{1}{2}\sin 2\theta(\sigma_r - \sigma_\theta) + \cos 2\theta\tau_{r\theta} \\ \frac{1}{2}\sin 2\theta(\sigma_r - \sigma_\theta) + \cos 2\theta\tau_{r\theta} & \sin^2\theta\sigma_r + \cos^2\theta\sigma_\theta + \sin 2\theta\tau_{r\theta} \end{bmatrix}$$



For $\psi = -\frac{p}{2\pi} r^2 \theta : \ \sigma_r = -\frac{p}{\pi}\theta, \ \sigma_\theta = -\frac{p}{\pi}\theta, \ \tau_{r\theta} = \frac{p}{2\pi}$

$$\Rightarrow \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \frac{p}{2\pi} \begin{bmatrix} -2\theta - \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -2\theta + \sin 2\theta \end{bmatrix}$$

- **Back to the present problem:** $\boxed{\psi = -\frac{p}{2\pi}\left(r_2^2\theta_2 - r_1^2\theta_1\right)}$

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \frac{p}{2\pi} \begin{bmatrix} -2(\theta_2 - \theta_1) - (\sin 2\theta_2 - \sin 2\theta_1) & \cos 2\theta_2 - \cos 2\theta_1 \\ \cos 2\theta_2 - \cos 2\theta_1 & -2(\theta_2 - \theta_1) + (\sin 2\theta_2 - \sin 2\theta_1) \end{bmatrix}$$

For $y = 0, x > a : \theta_2 = 0, \theta_1 = 0 : \ \sigma_y = 0$

For $y = 0, a > x > -a : \theta_2 = \pi, \theta_1 = 0 : \ \sigma_y = -p$

For $y = 0, x < -a : \theta_2 = \pi, \theta_1 = \pi : \ \sigma_y = 0$

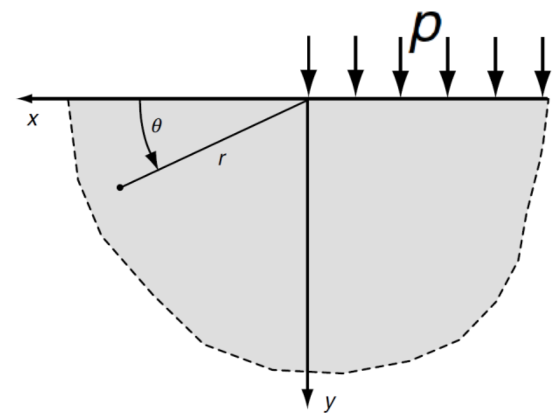
Only stress components in RCC can be added.

Half-Plane: Concentrated Normal Force

- Distributed pressure over $x < 0$

$$\psi = \psi(x,y) = a_2 r^2 - a_2 r^2 \cos 2\theta - \frac{p}{2\pi} r^2 \theta + \frac{p}{4\pi} r^2 \sin 2\theta$$

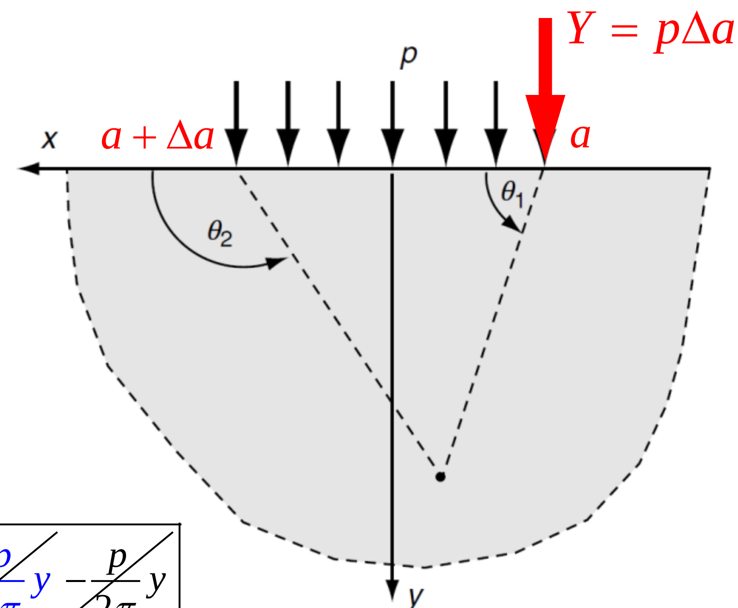
$$= a_2\left(x^2+y^2\right) - a_2\left(x^2-y^2\right) - \frac{p}{2\pi}\left(x^2+y^2\right)\tan^{-1}\frac{y}{x} + \frac{p}{2\pi}xy$$



- Distributed pressure over $a < x < a + \Delta a$

$$\psi = \psi(x,y,a+\mathrm{d}a) - \psi(x,y,a)$$

$$\boxed{\begin{aligned}\psi(x,y,a) &= a_2\left((x-a)^2+y^2\right) - a_2\left((x-a)^2-y^2\right)\\ &\quad - \frac{p}{2\pi}\left((x-a)^2+y^2\right)\tan^{-1}\frac{y}{x-a} + \frac{p}{2\pi}(x-a)y\end{aligned}}$$



- Concentrated normal force

$$\psi = \lim_{\Delta a \to 0}\left[\psi(x,y,a+\Delta a) - \psi(x,y,a)\right] = \Delta a \frac{\partial \psi(x,y,a)}{\partial a}$$

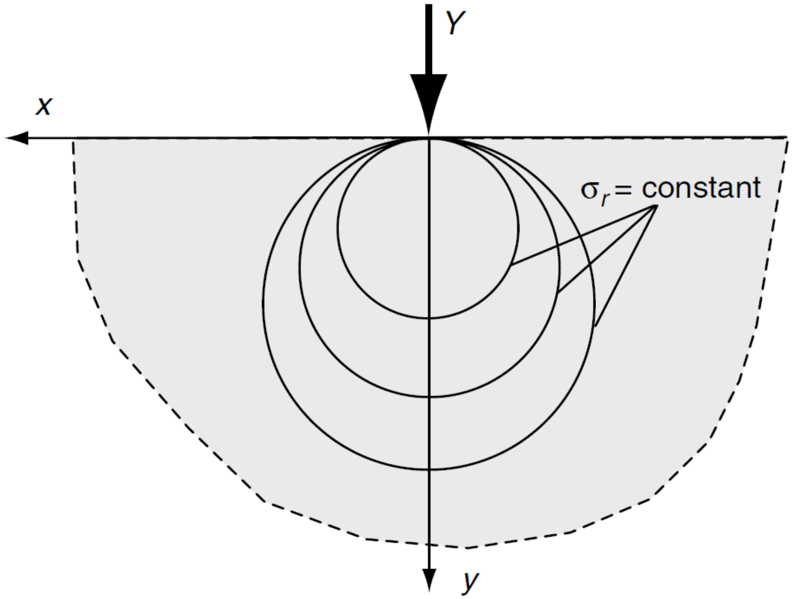
$$\boxed{\frac{\partial \psi(x,y,a)}{\partial a} = \cancel{-2a_2(x-a)} \cancel{+2a_2(x-a)} + \frac{p}{\pi}(x-a)\tan^{-1}\frac{y}{x-a} \cancel{-\frac{p}{2\pi}y} \cancel{-\frac{p}{2\pi}y}}$$

$$\Rightarrow \psi = \Delta a \frac{\partial \phi(x,y,a)}{\partial a} = \frac{p\Delta a}{\pi}(x-a)\tan^{-1}\frac{y}{x-a} = \frac{Y}{\pi}(x-a)\tan^{-1}\frac{y}{x-a}$$

Half-Plane: Concentrated Normal Force

- Back to the origin

$$a = 0 \;\Rightarrow\; \boxed{\psi = \frac{Y}{\pi} x \tan^{-1}\frac{y}{x} = \frac{Y}{\pi} r\theta\cos\theta}$$



- Stress field in polar coordinates

$$\sigma_r = -\frac{2Y}{\pi}\frac{1}{r}\sin\theta, \quad \sigma_\theta = \tau_{r\theta} = 0$$

- Stress field in RCC

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \cos^2\theta\sigma_r & \frac{1}{2}\sin 2\theta\sigma_r \\ \frac{1}{2}\sin 2\theta\sigma_r & \sin^2\theta\sigma_r \end{bmatrix} = -\frac{2Y}{\pi}\frac{1}{r}\begin{bmatrix} \cos^2\theta\sin\theta & \frac{1}{2}\sin 2\theta\sin\theta \\ \frac{1}{2}\sin 2\theta\sin\theta & \sin^2\theta\sin\theta \end{bmatrix} = -\frac{2Y}{\pi}\frac{1}{r^4}\begin{bmatrix} x^2 y & xy^2 \\ xy^2 & y^3 \end{bmatrix}$$

- Displacement field

$$u_r = \frac{1}{2}\left[(\kappa-1)\theta\cos\theta - (\kappa+1)\ln r\sin\theta + \sin\theta\right]$$

$$u_\theta = \frac{1}{2}\left[-(\kappa-1)\theta\sin\theta - (\kappa+1)\ln r\cos\theta - \cos\theta\right]$$

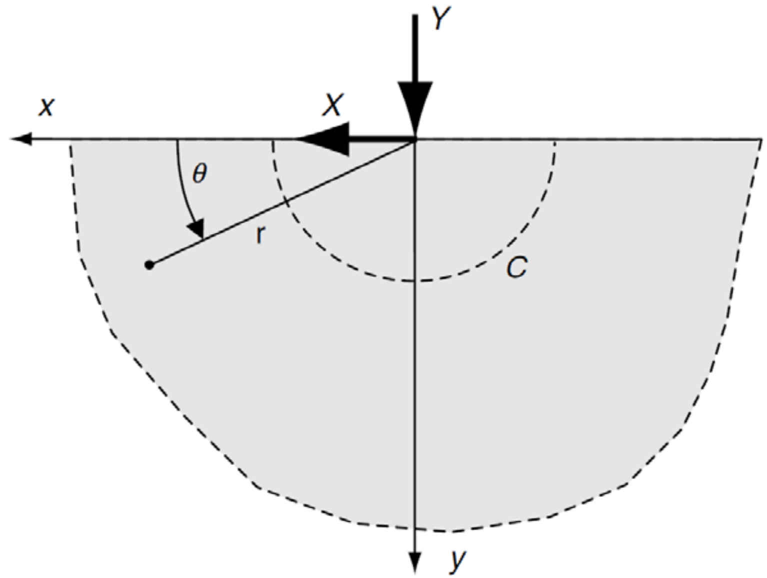
Half-Plane: Flamant Solution

- Conventional BCs

$$\sigma_\theta(r,0)=\tau_{r\theta}(r,0)=0$$
$$\tau_{r\theta}(r,\pi)=0,\ \sigma_\theta(r,\pi)=0$$

- Static equilibrium suggests

$$\int_0^\pi\left(\sigma_r r d\theta \sin\theta+\tau_{r\theta} r d\theta\cos\theta\right)+Y=0$$
$$\int_0^\pi\left(\sigma_r r d\theta \cos\theta-\tau_{r\theta} r d\theta\sin\theta\right)+X=0$$



- The above relations hold for an arbitrary *r*, thus: $\sigma_{ij}\sim X/r, Y/r$.
- From the general Michell solution

$$\sigma_r=a_{12}\frac{1}{r}\cos\theta-2a_{15}\frac{1}{r}\sin\theta+b_{12}\frac{1}{r}\sin\theta+2b_{15}\frac{1}{r}\cos\theta$$
$$\tau_{r\theta}=a_{12}\frac{1}{r}\sin\theta-b_{12}\frac{1}{r}\cos\theta$$
$$\sigma_\theta=a_{12}\frac{1}{r}\cos\theta+b_{12}\frac{1}{r}\sin\theta$$
$$\psi=\left(a_{12}r\ln r+a_{15}r\theta\right)\cos\theta+\left(b_{12}r\ln r+b_{15}r\theta\right)\sin\theta$$

Half-Plane: Flamant Solution

- Applying the BCs

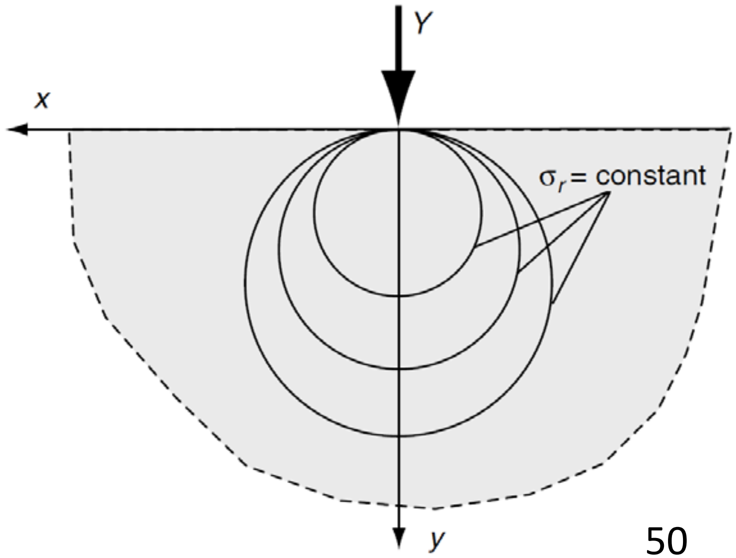
$$\begin{cases} \sigma_\theta(r,0)=0 \ \Rightarrow \ a_{12}=0 \\ \tau_{r\theta}(r,0)=0 \ \Rightarrow \ b_{12}=0 \\ \tau_{r\theta}(r,\pi)=0,\ \sigma_\theta(r,\pi)=0 \quad \text{satisfied.} \end{cases} \Rightarrow \begin{cases} \sigma_r = -2a_{15}\dfrac{1}{r}\sin\theta + 2b_{15}\dfrac{1}{r}\cos\theta \\ \tau_{r\theta}=\sigma_\theta=0 \\ \psi = a_{15} r\theta\cos\theta + b_{15} r\theta\sin\theta \end{cases}$$

$$\begin{cases} \int_0^\pi \sigma_r r\sin\theta d\theta + Y = 0 \\ \int_0^\pi \sigma_r r\cos\theta d\theta + X = 0 \end{cases} \Rightarrow \begin{cases} a_{15} = \dfrac{Y}{\pi} \\ b_{15} = -\dfrac{X}{\pi} \end{cases} \Rightarrow \begin{cases} \sigma_r = -\dfrac{2Y}{\pi}\dfrac{1}{r}\sin\theta - \dfrac{2X}{\pi}\dfrac{1}{r}\cos\theta \\ \tau_{r\theta}=\sigma_\theta=0 \\ \psi = \dfrac{Y}{\pi} r\theta\cos\theta - \dfrac{X}{\pi} r\theta\sin\theta \end{cases}$$

- Contours of constant radial stresses are circles that are tangent to the half-plane boundary at the loading point.

$$\sigma_r = -\frac{2Y}{\pi}\frac{1}{r}\sin\theta = -\frac{2Y}{\pi}\frac{y}{x^2+y^2}$$

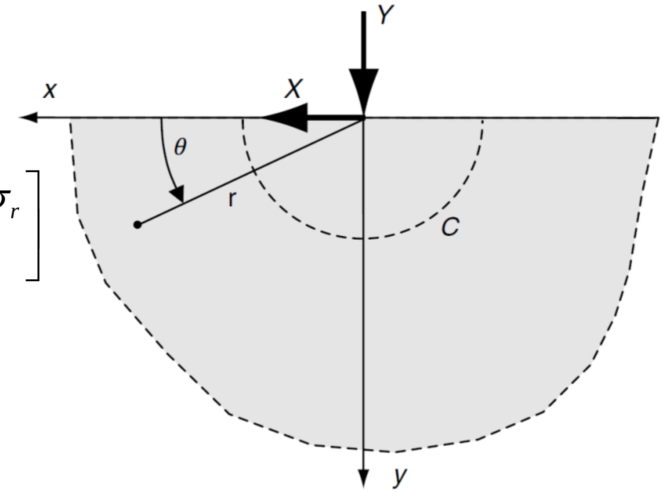
$$\Rightarrow \boxed{x^2 + \left(y + \frac{Y}{\pi\sigma_r}\right)^2 = \frac{Y^2}{\pi^2\sigma_r^2}}$$



Half-Plane: Generalized Superposition Method

- Concentrated forces

$$\begin{cases} \sigma_r = -\dfrac{2Y}{\pi}\dfrac{1}{r}\sin\theta - \dfrac{2X}{\pi}\dfrac{1}{r}\cos\theta \\ \tau_{r\theta} = \sigma_\theta = 0 \\ \psi = \dfrac{Y}{\pi} r\theta\cos\theta - \dfrac{X}{\pi} r\theta\sin\theta \end{cases} \Rightarrow \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \cos^2\theta\sigma_r & \sin\theta\cos\theta\sigma_r \\ \sin\theta\cos\theta\sigma_r & \sin^2\theta\sigma_r \end{bmatrix}$$



$$= -\frac{2}{\pi}\frac{1}{r}\begin{bmatrix} Y\sin\theta\cos^2\theta + X\cos^3\theta & Y\sin^2\theta\cos\theta + X\sin\theta\cos^2\theta \\ Y\sin^2\theta\cos\theta + X\sin\theta\cos^2\theta & Y\sin^3\theta + X\sin^2\theta\cos\theta \end{bmatrix}$$

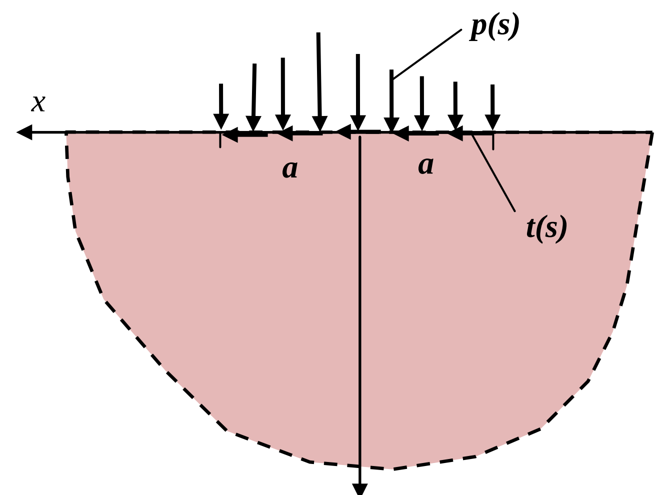
$$= -\frac{2}{\pi}\frac{1}{r^4}\begin{bmatrix} Yx^2y + Xx^3 & Yxy^2 + Xx^2y \\ Yxy^2 + Xx^2y & Yy^3 + Xxy^2 \end{bmatrix}$$

- Arbitrarily distributed forces

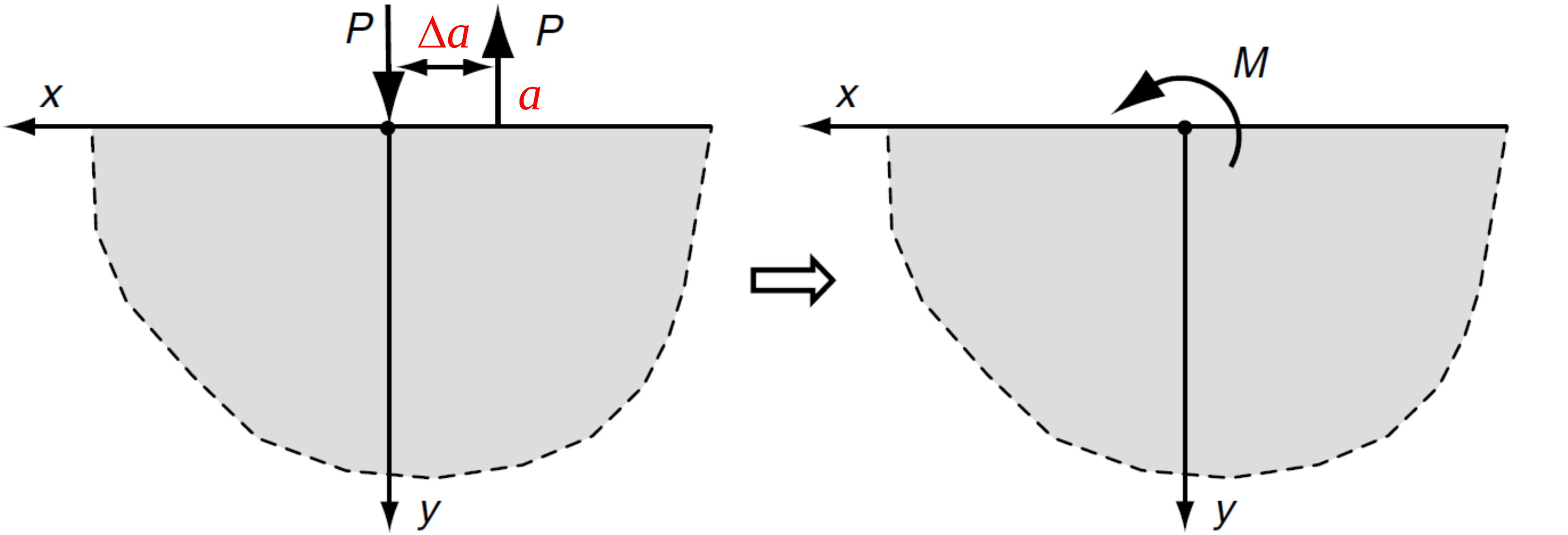
$$\sigma_x = -\frac{2}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)^2 y}{[(x-s)^2+y^2]^2}ds - \frac{2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^3}{[(x-s)^2+y^2]^2}ds$$

$$\sigma_y = -\frac{2}{\pi}\int_{-a}^{a}\frac{p(s)y^3}{[(x-s)^2+y^2]^2}ds - \frac{2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)y^2}{[(x-s)^2+y^2]^2}ds$$

$$\tau_{xy} = -\frac{2}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)y^2}{[(x-s)^2+y^2]^2}s - \frac{2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^2 y}{[(x-s)^2+y^2]^2}ds$$



Half-Plane: Concentrated Moment



- Method of superposition

$$\psi = \psi\left(x, y, a + \mathrm{d}a\right) - \psi\left(x, y, a\right) \Rightarrow \psi = \lim_{\Delta a \to 0}\left[\psi\left(x, y, a + \Delta a\right) - \psi\left(x, y, a\right)\right] = \Delta a \frac{\partial \psi\left(x, y, a\right)}{\partial a}$$

$$\psi = \psi\left(x, y, a\right) = \psi\left(x - a, y\right) = \frac{P}{\pi}\left(x - a\right)\tan^{-1}\frac{y}{x - a} \Rightarrow \frac{\partial \psi\left(x, y, a\right)}{\partial a} = -\frac{P}{\pi}\tan^{-1}\frac{y}{x - a} + \frac{P}{\pi}\frac{\left(x - a\right)y}{\left(x - a\right)^2 + y^2}$$

$$\Rightarrow \boxed{\psi = -\frac{P\Delta a}{\pi}\tan^{-1}\frac{y}{x - a} + \frac{P\Delta a}{\pi}\frac{\left(x - a\right)y}{\left(x - a\right)^2 + y^2}}$$

Half-Plane: Concentrated Moment

- Back to the origin

$$a = 0, \;\; P\Delta a = M$$

$$\Rightarrow \boxed{\psi = -\frac{M}{\pi}\tan^{-1}\frac{y}{x} + \frac{M}{\pi}\frac{xy}{x^2+y^2} = -\frac{M}{\pi}\theta + \frac{M}{2\pi}\sin 2\theta}$$

- Stress field

$$\sigma_r = -\frac{2M}{\pi}\frac{1}{r^2}\sin 2\theta, \;\; \tau_{r\theta} = -\frac{M}{\pi}\frac{1}{r^2}\left(1-\cos 2\theta\right), \;\; \sigma_\theta = 0$$

- Displacement field

$$u_r = \frac{\left(\kappa+1\right)M}{4\pi G}\frac{1}{r}\sin 2\theta + u_o\cos\theta + v_o\sin\theta$$

$$u_\theta = \frac{M}{4\pi G}\frac{1}{r}\left[2+\left(\kappa-1\right)\cos 2\theta\right] - u_o\sin\theta + v_o\cos\theta + \omega_o r$$

Outline

- Polar Coordinate Formulation
- Axisymmetric Solutions to Biharmonic Equations
- Cylinders under Boundary Pressures
- Hole in Infinite Media
- Pure Bending of Curved Beams
- Rotating Disk/Cylinder Problem
- General Solutions to Biharmonic equation
- Stress Concentration around a Hole
- Transverse Bending of Curved Beams
- Wedge Problems
- Quarter-Plane Problems
- Half-Plane Problems