Displacement and Strain

Outline

- Generalized Displacement
- Small Deformation Theory
- Continuum Motion & Deformation
- Strain & Rotation
- Principal Strains
- Spherical and Deviatoric Strain
- Cylindrical Strain and Rotation
- Spherical Strain and Rotation
- Strain Compatibility
- Domain Connectivity

Displacement

- **Concept of displacement:** coordinate difference of the same material point in two reference states.
- **Displacement**

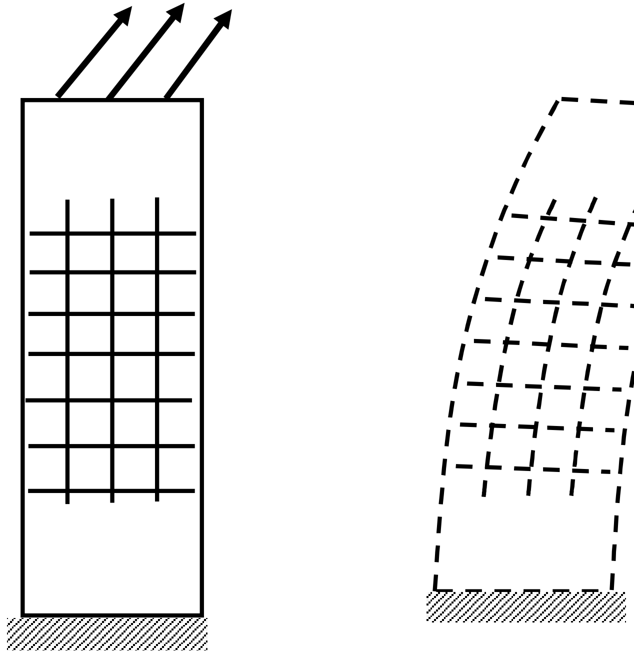
= Rigid-body translation

+ Rigid-body rotation

+ Strain deformation

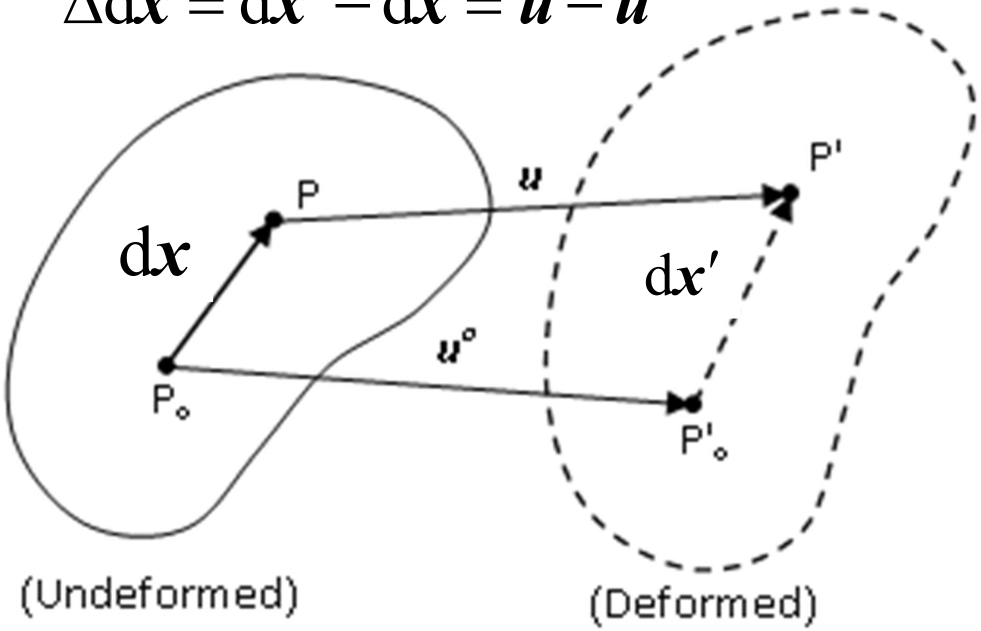
- Rigid-body motion: the distance and angle among all material points remain the same.
- Strain deformation: a material is said to be deformed or strained when the distance or angle among material points is changed.
- We are not concerned with rigid-body motions in elasticity theory.

Small Deformation Theory



(Undeformed) (Deformed)

$$\Delta \mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{x}' - \mathrm{d}\boldsymbol{x} = \boldsymbol{u} - \boldsymbol{u}^o$$



- Taylor expansion of $\boldsymbol{u}$ w.r.t. $\boldsymbol{u}^{\mathbf{o}}$:

$$\boldsymbol{u} = \boldsymbol{u}^{\boldsymbol{o}} + \boldsymbol{u}\overleftarrow{\nabla} \cdot \mathrm{d}\boldsymbol{x} + \cdots$$

$$u_i = u_i^o + u_{i,j}\mathrm{d}x_j + \cdots$$

$$\Rightarrow \boxed{\Delta \mathrm{d}x_i = u_i - u_i^o \approx u_{i,j}\mathrm{d}x_j}$$

$$u = u^o + \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z$$

$$v = v^o + \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y + \frac{\partial v}{\partial z}\mathrm{d}z$$

$$w = w^o + \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial y}\mathrm{d}y + \frac{\partial w}{\partial z}\mathrm{d}z$$

Small Deformation Theory

- Displacement gradient

$$u_{i,j} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) = \varepsilon_{ij} + \omega_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}); \quad \varepsilon = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} + \nabla\boldsymbol{u}\right), \text{ strain tensor (symmetric)}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}); \quad \omega = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} - \nabla\boldsymbol{u}\right), \text{ rotation tensor (anti-symmetric)}$$

- Total displacement

$$u_i = u_i^o + \left(\varepsilon_{ij} + \omega_{ij}\right)\mathrm{d}x_j + \cdots$$

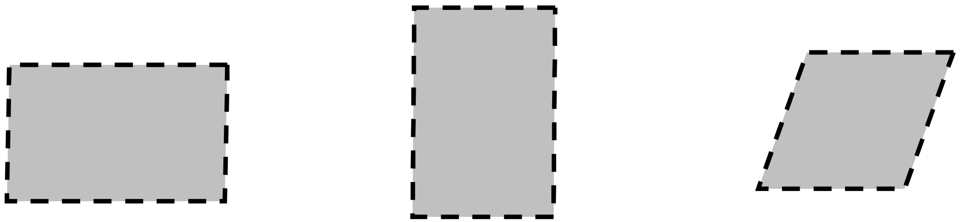
Continuum Rigid-body Motion & Deformation

- Components of total displacement at a material point

$$u_i = u_i^o + \varepsilon_{ij}\mathrm{d}x_j + \omega_{ij}\mathrm{d}x_j$$

General displacement (u_i); Rigid-body displacement (u_i^o); Strain displacement ($\varepsilon_{ij}\mathrm{d}x_j$); Rigid-body rotation ($\omega_{ij}\mathrm{d}x_j$)

(Undeformed Element) **(Rigid Body Rotation)**



(Horizontal Extension) **(Vertical Extension)** **(Shearing Deformation)**

Two-dimensional strain-displacement relation

$$B\left(x+\mathrm{d}x,y\right):$$

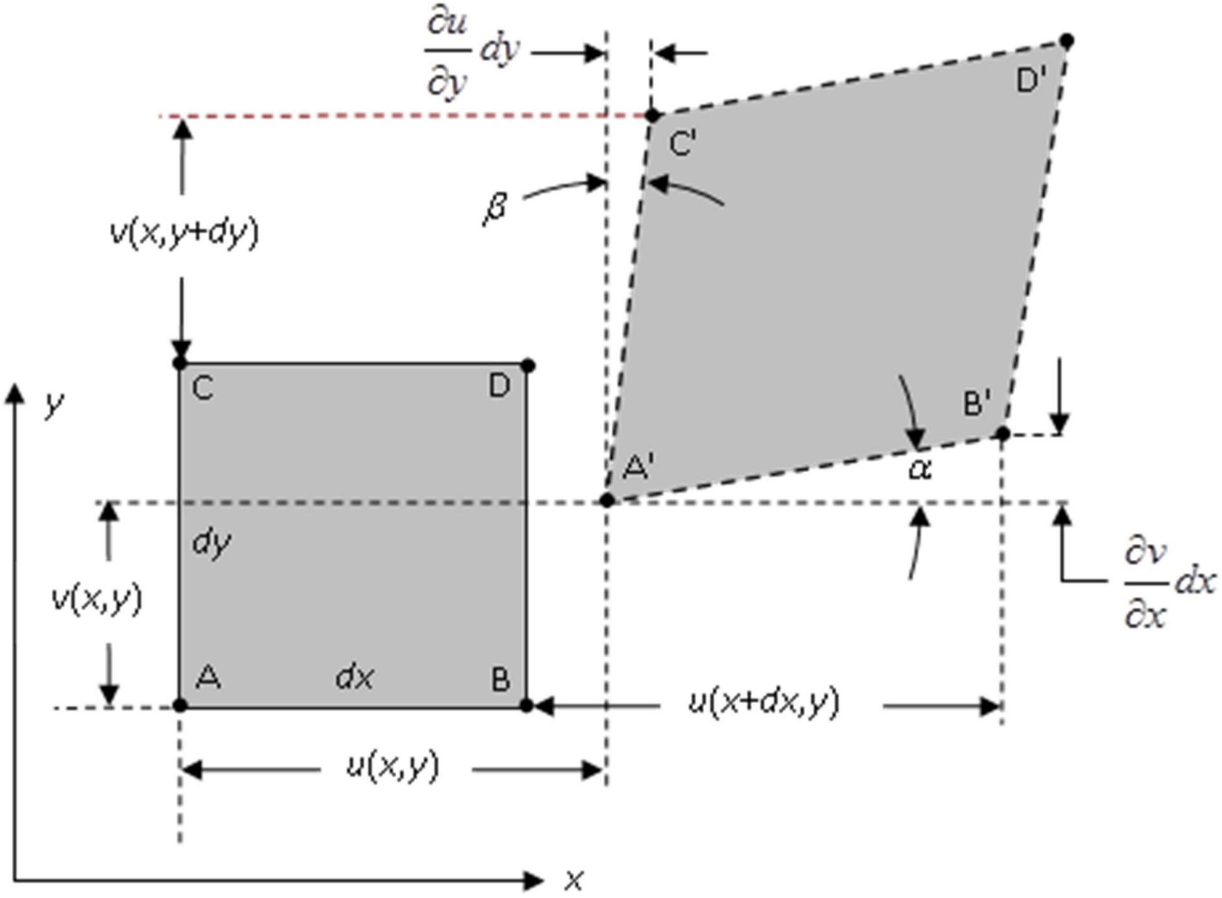
$$u\left(x+\mathrm{d}x,y\right)=u\left(x,y\right)+\frac{\partial u}{\partial x}\mathrm{d}x,$$

$$v\left(x+\mathrm{d}x,y\right)=v\left(x,y\right)+\frac{\partial v}{\partial x}\mathrm{d}x;$$

$$C\left(x,y+\mathrm{d}y\right):$$

$$u\left(x,y+\mathrm{d}y\right)=u\left(x,y\right)+\frac{\partial u}{\partial y}\mathrm{d}y,$$

$$v\left(x,y+dy\right)=v\left(x,y\right)+\frac{\partial v}{\partial y}dy.$$



$$A'B'=\sqrt{\left(\mathrm{d}x+\frac{\partial u}{\partial x}\mathrm{d}x\right)^2+\left(\frac{\partial v}{\partial x}\mathrm{d}x\right)^2}=\mathrm{d}x\sqrt{1+2\frac{\partial u}{\partial x}+\cancel{\left(\frac{\partial u}{\partial x}\right)^2}+\cancel{\left(\frac{\partial v}{\partial x}\right)^2}}\approx\left(1+\frac{\partial u}{\partial x}\right)\mathrm{d}x;$$

$$\alpha\approx\tan\alpha=\frac{\partial v}{\partial x}\mathrm{d}x\Big/\left(\mathrm{d}x+\frac{\partial u}{\partial x}\mathrm{d}x\right)\approx\frac{\partial v}{\partial x},\quad \beta\approx\tan\beta=\frac{\partial u}{\partial y}\mathrm{d}y\Big/\left(\mathrm{d}y+\frac{\partial v}{\partial y}\mathrm{d}y\right)\approx\frac{\partial u}{\partial y}.$$

Two-dimensional Geometric Deformation

- Normal strain

$$\varepsilon_x = \varepsilon_{xx} = \frac{A'B' - AB}{AB} = \frac{\left(1+\frac{\partial u}{\partial x}\right)\mathrm{d}x - \mathrm{d}x}{\mathrm{d}x} = \frac{\partial u}{\partial x},$$

$$\varepsilon_y = \varepsilon_{yy} = \frac{A'C' - AC}{AC} = \frac{\left(1+\frac{\partial v}{\partial y}\right)\mathrm{d}y - \mathrm{d}y}{\mathrm{d}y} = \frac{\partial v}{\partial y}.$$

- Engineering shear strain

$$\gamma_{xy} = \frac{\pi}{2} - \angle C'A'B' = \alpha + \beta = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y};$$

- Shear strain

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

- 3-D Strain-displacement relationship

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

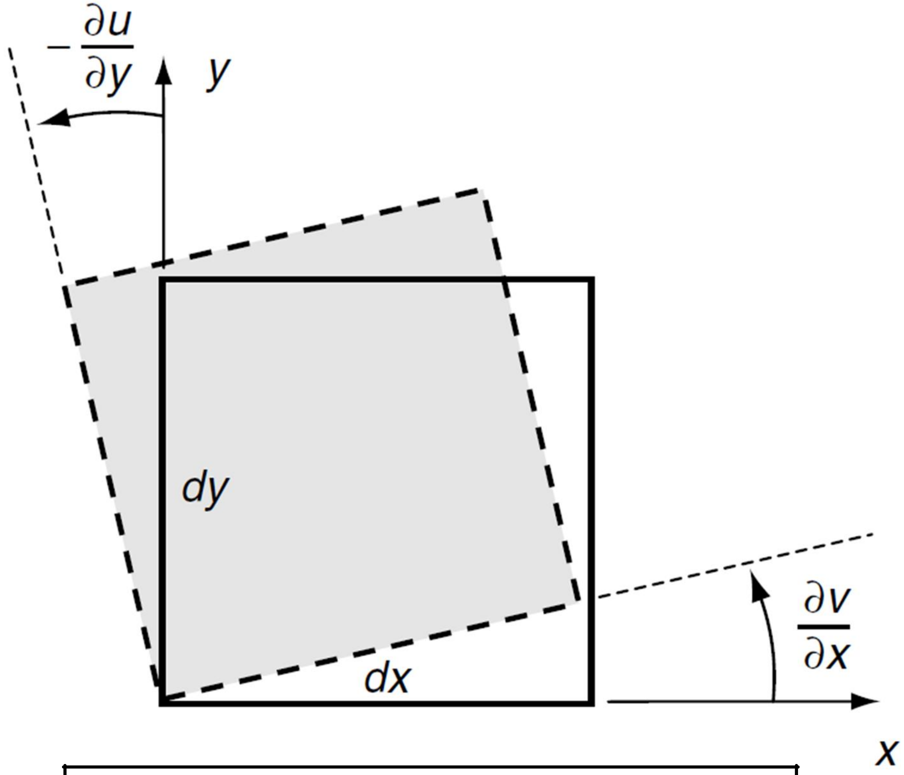
Two-dimensional Rigid-body Rotation

- Rigid-body rotation around z-axis

$$\omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- Integrate for constant rotation

$$\Rightarrow \begin{cases} u^* = u^o - \omega_z y \\ v^* = v^o + \omega_z x \end{cases}$$



- 3-D rigid-body rotation

$$\boxed{\omega_i = -1/2\,\varepsilon_{ijk}\omega_{jk} = 1/2\,\varepsilon_{ijk}u_{k,j}}$$

$$\begin{cases} u^* = u^o - \omega_z y + \omega_y z \\ v^* = v^o - \omega_x z + \omega_z x \\ w^* = w^o - \omega_y x + \omega_x y \end{cases}$$

$$\omega_1 = \omega_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

$$\omega_2 = \omega_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$

$$\omega_3 = \omega_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

Sample Problem

Determine the displacement gradient, strain and rotation tensors for the following displacement field: $u = Ax^2 y$, $v = Byz$, $w = Cxz^3$, where *A, B,* and *C* are arbitrary constants. Also calculate the dual rotation vector $\boldsymbol{\omega} = (1/2)(\nabla \times \boldsymbol{u})$.

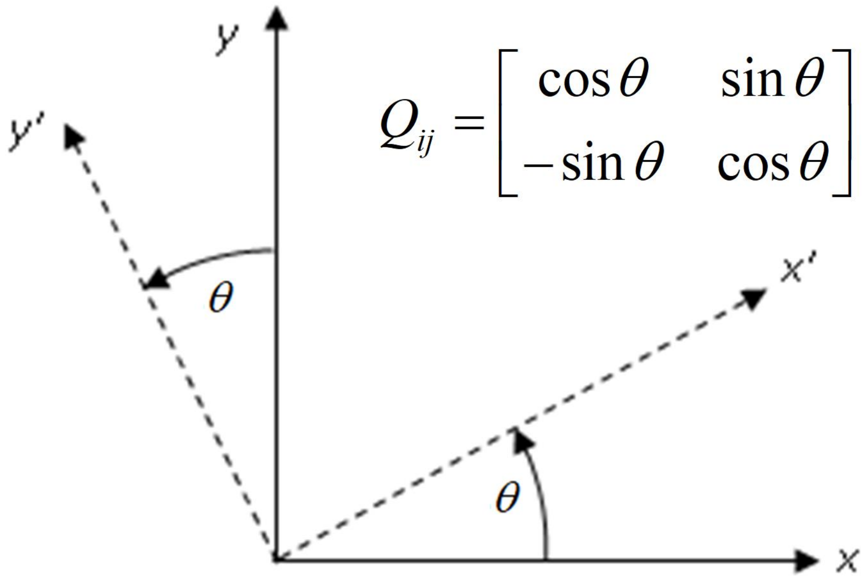
$$u_{i,j} = \begin{bmatrix} 2Axy & Ax^2 & 0 \\ 0 & Bz & By \\ Cz^3 & 0 & 3Cxz^2 \end{bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \begin{bmatrix} 2Axy & Ax^2/2 & Cz^3/2 \\ Ax^2/2 & Bz & By/2 \\ Cz^3/2 & By/2 & 3Cxz^2 \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = \begin{bmatrix} 0 & Ax^2/2 & -Cz^3/2 \\ -Ax^2/2 & 0 & By/2 \\ Cz^3/2 & -By/2 & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} = (1/2)(\nabla \times \boldsymbol{u}) = \frac{1}{2}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Ax^2 y & Byz & Cxz^3 \end{vmatrix} = \frac{1}{2}\left(-By\boldsymbol{e}_1 - Cz^3\boldsymbol{e}_2 - Ax^2\boldsymbol{e}_3\right)$$

2-D Principal Strains by Transformation



$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\varepsilon'_{\alpha\beta} = Q_{\alpha\delta} Q_{\beta\gamma} \varepsilon_{\delta\gamma}$$

$$\Rightarrow \begin{cases} \varepsilon'_{11} = Q_{1\delta} Q_{1\gamma} \varepsilon_{\delta\gamma} \\ \dots \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon'_x = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + 2\varepsilon_{xy} \sin\theta\cos\theta \\ \varepsilon'_y = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - 2\varepsilon_{xy} \sin\theta\cos\theta \\ \varepsilon'_{xy} = -\varepsilon_x \sin\theta\cos\theta + \varepsilon_y \sin\theta\cos\theta + \varepsilon_{xy}(\cos^2\theta - \sin^2\theta) \end{cases}$$

⇩

$$\begin{cases} \varepsilon'_x = \dfrac{\varepsilon_x + \varepsilon_y}{2} + \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon'_y = \dfrac{\varepsilon_x + \varepsilon_y}{2} - \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \varepsilon_{xy} \sin 2\theta \\ \varepsilon'_{xy} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

⇦

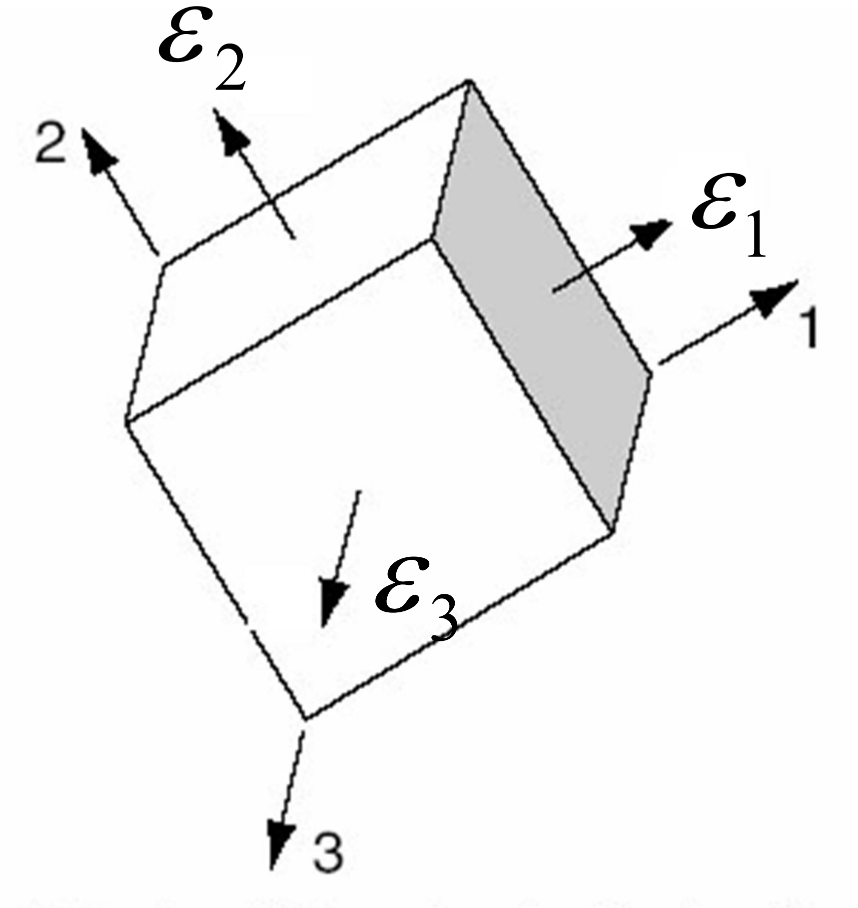
$$\left(\varepsilon'_x - \varepsilon_{\text{ave}}\right)^2 + \varepsilon'^2_{xy} = R^2$$

$$\varepsilon_{\text{ave}} = \frac{\varepsilon_x + \varepsilon_y}{2}; \quad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2}$$

⇩ ⇩

$$\varepsilon'_x + \varepsilon'_y = \varepsilon_x + \varepsilon_y; \qquad \varepsilon'_{\max,\min} = \varepsilon_{\text{ave}} \pm R; \qquad \tan 2\theta_p = \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y}$$

3-D Principal Strains by Eigen-equation



(Principal Coordinate System)

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

$$\varepsilon_{ij} n_j = \varepsilon_n n_i$$

$$\det\left[\varepsilon_{ij} - \varepsilon_n \delta_{ij}\right] = 0$$

$$\boxed{-\varepsilon_n{}^3 + I_1 \varepsilon_n{}^2 - I_2 \varepsilon_n + I_3 = 0}$$

$$I_1 = \varepsilon_{kk}$$

$$I_2 = \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}\right)$$

$$I_3 = \det\left[\varepsilon_{ij}\right]$$

Spherical and Deviatoric Strain

- Decomposition of the strain tensor

$$\varepsilon_{ij} = \tilde{\varepsilon}_{ij} + \hat{\varepsilon}_{ij}$$

- Spherical (mean) strain tensor: volumetric deformation + isotropic

$$\tilde{\varepsilon}_{ij} = \varepsilon_m \delta_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1}{3}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\delta_{ij}$$

- Deviatoric (octahedral) strain tensor: shape change

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \tilde{\varepsilon}_{ij}$$

- Relationships among principal strains and directions

$$\hat{\varepsilon}_{ij}\hat{n}_j = \hat{\varepsilon}_n \hat{n}_i \Rightarrow \left(\varepsilon_{ij} - \varepsilon_m \delta_{ij}\right)\hat{n}_j = \hat{\varepsilon}_n \hat{n}_i \Rightarrow \varepsilon_{ij}\hat{n}_j = \left(\hat{\varepsilon}_n + \varepsilon_m\right)\hat{n}_i$$

$$\varepsilon_{ij} n_j = \varepsilon_n n_i \Rightarrow \begin{cases} \hat{n}_i = n_i \\ \hat{\varepsilon}_n = \varepsilon_n - \varepsilon_m \end{cases}$$

Cylindrical Strain and Rotation

$$\varepsilon = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} + \nabla\boldsymbol{u}\right); \quad \omega = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} - \nabla\boldsymbol{u}\right); \ \boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z;$$

$$\boldsymbol{u}\overleftarrow{\nabla} = \begin{bmatrix} \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial u_r}{\partial z}\boldsymbol{e}_r\boldsymbol{e}_z + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta \\ + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_\theta\boldsymbol{e}_z + \frac{\partial u_z}{\partial r}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z \end{bmatrix}$$

$$\boxed{\begin{aligned} &\omega_r = \omega_\theta = \omega_z = 0, \omega_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r}\right), \\ &\omega_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \omega_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\right); \end{aligned}}$$

$$\boxed{\begin{aligned} &\varepsilon_r = \frac{\partial u_r}{\partial r}, \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \varepsilon_z = \frac{\partial u_z}{\partial z}, \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right), \\ &\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right). \end{aligned}}$$

Spherical Strain and Rotation

$$\varepsilon = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} + \nabla \boldsymbol{u}\right); \quad \omega = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} - \nabla \boldsymbol{u}\right); \ \boldsymbol{u} = u_R \boldsymbol{e}_R + u_\varphi \boldsymbol{e}_\varphi + u_\theta \boldsymbol{e}_\theta;$$

$$\boldsymbol{u}\overleftarrow{\nabla} = \begin{bmatrix} \dfrac{\partial u_R}{\partial R}\boldsymbol{e}_R\boldsymbol{e}_R + \left(\dfrac{1}{R}\dfrac{\partial u_R}{\partial \varphi} - \dfrac{u_\varphi}{R}\right)\boldsymbol{e}_R\boldsymbol{e}_\varphi + \left(\dfrac{1}{R\sin\varphi}\dfrac{\partial u_R}{\partial \theta} - \dfrac{u_\theta}{R}\right)\boldsymbol{e}_R\boldsymbol{e}_\theta \\ +\dfrac{\partial u_\varphi}{\partial R}\boldsymbol{e}_\varphi\boldsymbol{e}_R + \left(\dfrac{u_R}{R} + \dfrac{1}{R}\dfrac{\partial u_\varphi}{\partial \varphi}\right)\boldsymbol{e}_\varphi\boldsymbol{e}_\varphi + \left(-\dfrac{\cot\varphi u_\theta}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\varphi}{\partial \theta}\right)\boldsymbol{e}_\varphi\boldsymbol{e}_\theta \\ +\dfrac{\partial u_\theta}{\partial R}\boldsymbol{e}_\theta\boldsymbol{e}_R + \dfrac{1}{R}\dfrac{\partial u_\theta}{\partial \varphi}\boldsymbol{e}_\theta\boldsymbol{e}_\varphi + \left(\dfrac{u_R}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{\cot\varphi u_\varphi}{R}\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta \end{bmatrix}$$

$$\omega_R = \omega_\theta = \omega_\varphi = 0, \omega_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R} - \frac{\partial u_\theta}{\partial R}\right),$$

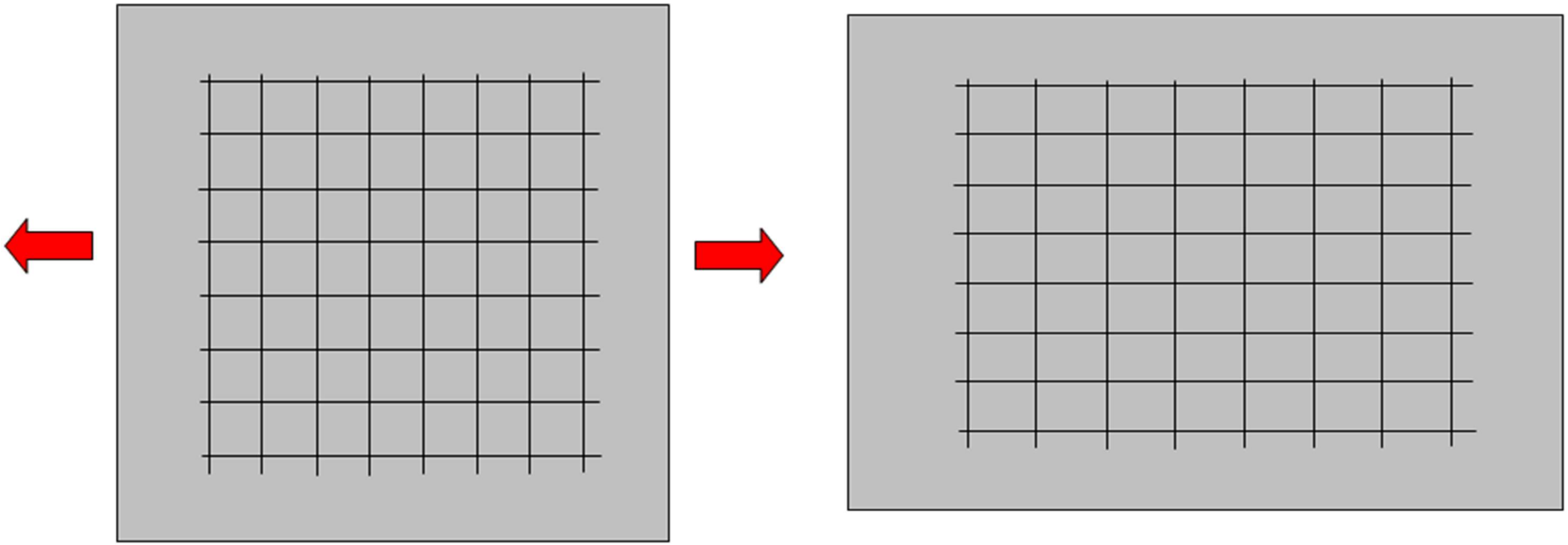
$$\omega_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial \varphi} + \frac{\cot\varphi u_\theta}{R} - \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial \theta}\right), \omega_{\varphi R} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial R} - \frac{1}{R}\frac{\partial u_R}{\partial \varphi} + \frac{u_\varphi}{R}\right);$$

$$\varepsilon_R = \frac{\partial u_R}{\partial R}, \varepsilon_\varphi = \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial \varphi}, \varepsilon_\theta = \frac{u_R}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial \theta} + \frac{\cot\varphi u_\varphi}{R}, \varepsilon_{R\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial R} + \frac{1}{R}\frac{\partial u_R}{\partial \varphi} - \frac{u_\varphi}{R}\right),$$

$$\varepsilon_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R} + \frac{\partial u_\theta}{\partial R}\right), \varepsilon_{\varphi\theta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial \varphi} - \frac{\cot\varphi u_\theta}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial \theta}\right).$$

Strain Compatibility - Concept

- Normally we want continuous and single-valued displacements; i.e. a mesh that fits perfectly together after deformation.



Undeformed State

Deformed State

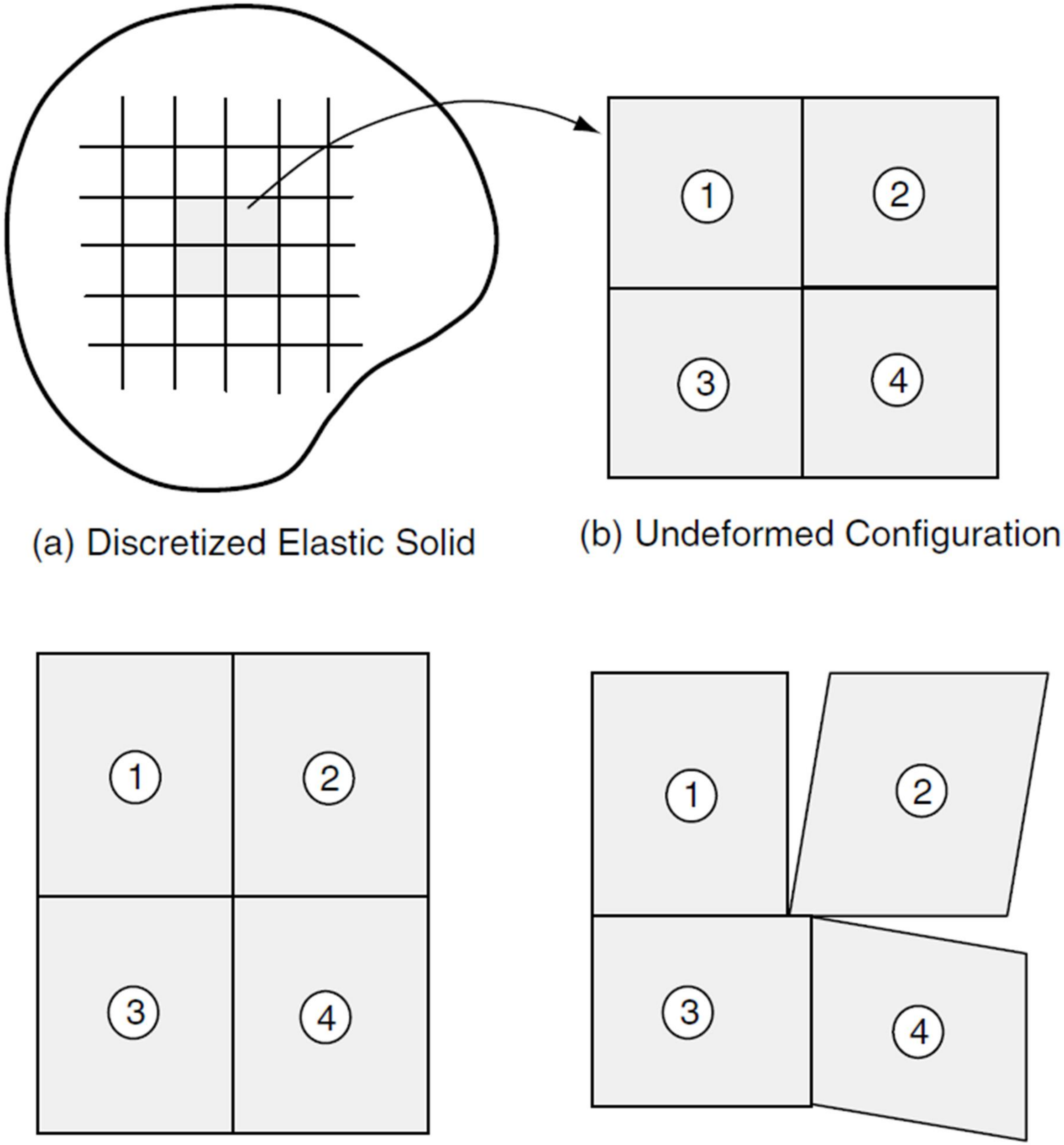
Strain Compatibility – Mathematical Context

- Strain-displacement relationship

$$\varepsilon_x = \frac{\partial u}{\partial x}, \; \varepsilon_y = \frac{\partial v}{\partial y}, \; \varepsilon_z = \frac{\partial w}{\partial z}, \; \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \; \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \; \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

- Given the three displacements: We have six equations to easily determine the six strain components.
- Given the six strains: We have six equations to determine three displacement components. This is an over-determined system and in general will not yield continuous single-valued displacements unless the strain components satisfy some additional relations.

Strain Compatibility – Physical Interpretation



(a) Discretized Elastic Solid (b) Undeformed Configuration

(c) Deformed Configuration Continuous Displacements (d) Deformed Configuration Discontinuous Displacements

Compatibility Equations

- Differentiating twice the strain-displacement relationship

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\Rightarrow\begin{cases}\varepsilon_{ij,kl}=\frac{1}{2}\left(u_{i,jkl}+u_{j,ikl}\right);\ \ \varepsilon_{kl,ij}=\frac{1}{2}\left(u_{k,lij}+u_{l,kij}\right);\\ \varepsilon_{ik,jl}=\frac{1}{2}\left(u_{i,kjl}+u_{k,ijl}\right);\ \ \varepsilon_{jl,ik}=\frac{1}{2}\left(u_{j,lik}+u_{l,jik}\right).\end{cases}$$

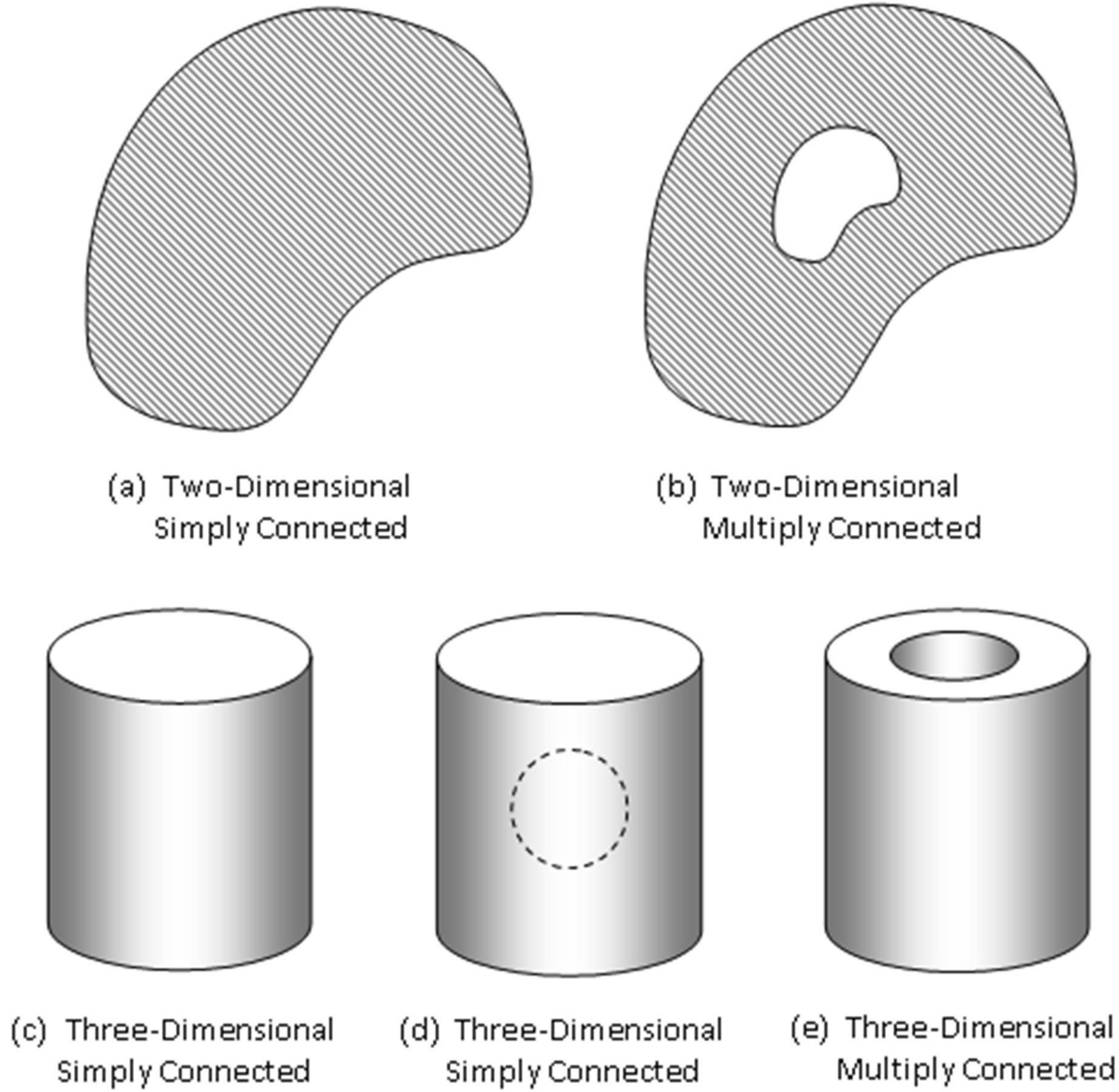
- The continuity of displacements implies the interchangeability of partial derivatives

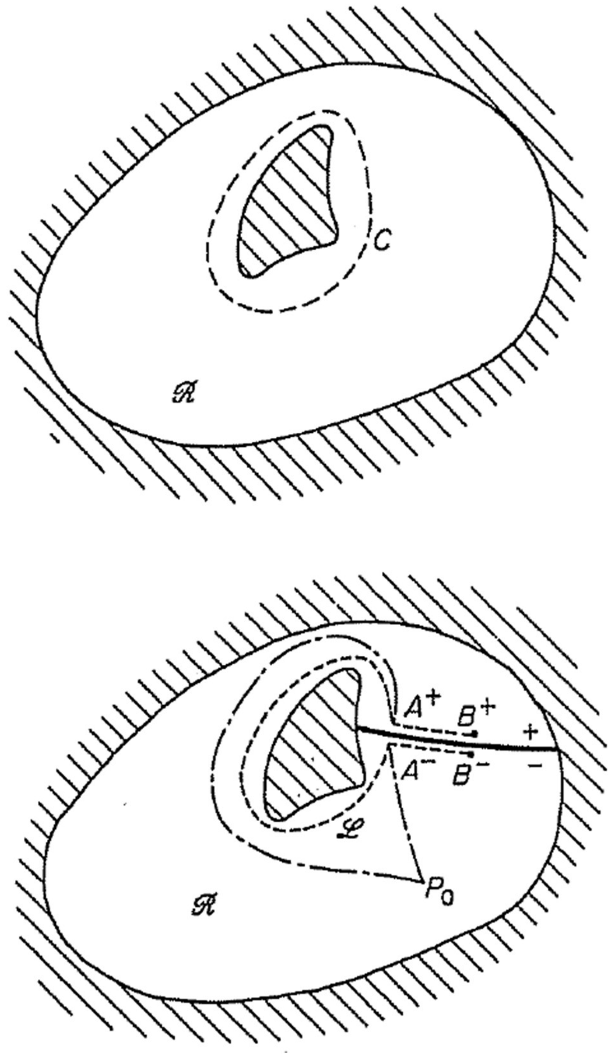
$$\Rightarrow\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0$$

- This **strain compatibility condition** forms the **necessary and sufficient condition** for continuous and single-valuedness displacements (up to a rigid-body motion) in **simply connected regions**.
- For multiply connected regions, strain compatibility is necessary but no longer sufficient. Additional conditions must be imposed.

Domain Connectivity

- Simply connected: all simple closed curves drawn in the region can be continuously shrunk to a point without going outside the region.





Compatibility Equations

- In 2-D, only 1 out of the 16 equations is meaningful and independent.

$$\left.\begin{array}{l}\varepsilon'_{ijkl} = \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \\ \varepsilon'_{ijkl} = \varepsilon'_{jikl} = \varepsilon'_{ijlk} = \varepsilon'_{klij} \\ i,j,k,l = 1,2\end{array}\right\} \Rightarrow \begin{bmatrix} \varepsilon'_{1111} & \varepsilon'_{1122} & \varepsilon'_{1112} \\ & \varepsilon'_{2222} & \varepsilon'_{2212} \\ \text{symm.} & & \varepsilon'_{1212}\end{bmatrix}$$

$$\varepsilon'_{1111} = \varepsilon'_{2222} = 0;$$

$$\varepsilon'_{1122} = \varepsilon_{11,22} + \varepsilon_{22,11} - \varepsilon_{12,12} - \varepsilon_{12,12}; \ \varepsilon'_{1112} = \varepsilon_{11,12} + \varepsilon_{12,11} - \varepsilon_{11,12} - \varepsilon_{12,11} = 0;$$

$$\varepsilon'_{1212} = \varepsilon_{12,12} + \varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = \varepsilon'_{1122}; \ \varepsilon'_{2212} = \varepsilon_{22,12} + \varepsilon_{12,22} - \varepsilon_{21,22} - \varepsilon_{22,21} = 0.$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}}$$

Compatibility Equations

- In 3-D:

$$\left.\begin{aligned} &\varepsilon'_{ijkl} = \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \\ &\varepsilon'_{ijkl} = \varepsilon'_{jikl} = \varepsilon'_{ijlk} = \varepsilon'_{klij} \\ &i, j, k, l = 1,2,3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} \varepsilon'_{1111} & \varepsilon'_{1122} & \varepsilon'_{1133} & \varepsilon'_{1112} & \varepsilon'_{1113} & \varepsilon'_{1123} \\ & \varepsilon'_{2222} & \varepsilon'_{2233} & \varepsilon'_{2212} & \varepsilon'_{2213} & \varepsilon'_{2223} \\ & & \varepsilon'_{3333} & \varepsilon'_{3312} & \varepsilon'_{3313} & \varepsilon'_{3323} \\ & & & \varepsilon'_{1212} & \varepsilon'_{1213} & \varepsilon'_{1223} \\ & \text{symm.} & & & \varepsilon'_{1313} & \varepsilon'_{1323} \\ & & & & & \varepsilon'_{2323} \end{bmatrix}$$

$$\varepsilon'_{1111} = \varepsilon'_{2222} = \varepsilon'_{3333} = 0;$$

$$\varepsilon'_{1122} = \varepsilon_{11,22} + \varepsilon_{22,11} - \varepsilon_{12,12} - \varepsilon_{12,12}; \ \varepsilon'_{1133} = \varepsilon_{11,33} + \varepsilon_{33,11} - \varepsilon_{13,13} - \varepsilon_{13,13};$$

$$\varepsilon'_{2233} = \varepsilon_{22,33} + \varepsilon_{33,22} - \varepsilon_{23,23} - \varepsilon_{23,23}; \ \varepsilon'_{1112} = \varepsilon_{11,12} + \varepsilon_{12,11} - \varepsilon_{11,12} - \varepsilon_{12,11} = 0;$$

$$\varepsilon'_{1113} = \varepsilon_{11,13} + \varepsilon_{13,11} - \varepsilon_{11,13} - \varepsilon_{13,11} = 0; \ \varepsilon'_{1123} = \varepsilon_{11,23} + \varepsilon_{23,11} - \varepsilon_{12,13} - \varepsilon_{13,12};$$

$$\varepsilon'_{2212} = \varepsilon_{22,12} + \varepsilon_{12,22} - \varepsilon_{21,22} - \varepsilon_{22,21} = 0; \ \varepsilon'_{2213} = \varepsilon_{22,13} + \varepsilon_{13,22} - \varepsilon_{21,23} - \varepsilon_{23,21};$$

$$\varepsilon'_{2223} = \varepsilon_{22,23} + \varepsilon_{23,22} - \varepsilon_{22,23} - \varepsilon_{23,22} = 0; \ \varepsilon'_{3312} = \varepsilon_{33,12} + \varepsilon_{12,33} - \varepsilon_{31,32} - \varepsilon_{32,31};$$

$$\varepsilon'_{1212} = \varepsilon_{12,12} + \varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11}; \ \varepsilon'_{1313} = \varepsilon_{13,13} + \varepsilon_{13,13} - \varepsilon_{11,33} - \varepsilon_{33,11};$$

$$\varepsilon'_{2323} = \varepsilon_{23,23} + \varepsilon_{23,23} - \varepsilon_{22,33} - \varepsilon_{33,22}; \ \varepsilon'_{1213} = \varepsilon_{12,13} + \varepsilon_{13,12} - \varepsilon_{11,23} - \varepsilon_{23,11};$$

$$\varepsilon'_{1223} = \varepsilon_{12,23} + \varepsilon_{23,12} - \varepsilon_{12,23} - \varepsilon_{23,12}; \ \varepsilon'_{1323} = \varepsilon_{13,23} + \varepsilon_{23,13} - \varepsilon_{12,33} - \varepsilon_{33,12};$$

Compatibility Equations

- Only 6 out of the 81 are meaningful

$$\boxed{1}: \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z};$$

$$\boxed{2}: \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z};$$

$$\boxed{3}: \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y};$$

$$\boxed{4}: \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right);$$

$$\boxed{5}: \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right);$$

$$\boxed{6}: \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right).$$

- These 6 equations may be obtained by letting $k=l$

$$k = l \Rightarrow \boxed{\varepsilon_{ij,kk} + \varepsilon_{kk,ij} - \varepsilon_{ik,jk} - \varepsilon_{jk,ik} = 0}$$

$$\boxed{i = 1, j = 1} \Rightarrow \boxed{1'} = \boxed{2} + \boxed{3}:$$

$$\left(\cancel{\varepsilon_{11,11}} + \varepsilon_{11,22} + \varepsilon_{11,33}\right) + \left(\cancel{\varepsilon_{11,11}} + \varepsilon_{22,11} + \varepsilon_{33,11}\right)$$

$$-2\left(\cancel{\varepsilon_{11,11}} + \varepsilon_{12,12} + \varepsilon_{13,13}\right) = 0;$$

$$\boxed{i = 2, j = 2} \Rightarrow \boxed{2'} = \boxed{3} + \boxed{1};$$

$$\boxed{i = 3, j = 3} \Rightarrow \boxed{3'} = \boxed{1} + \boxed{2};$$

$$\boxed{i = 1, j = 2} \Rightarrow \boxed{4'} = \boxed{4}:$$

$$\left(\cancel{\varepsilon_{12,11}} + \cancel{\varepsilon_{12,22}} + \varepsilon_{12,33}\right) + \left(\cancel{\varepsilon_{11,12}} + \cancel{\varepsilon_{22,12}} + \varepsilon_{33,12}\right)$$

$$-\left(\cancel{\varepsilon_{11,21}} + \cancel{\varepsilon_{12,22}} + \varepsilon_{13,23}\right) - \left(\cancel{\varepsilon_{21,11}} + \cancel{\varepsilon_{22,12}} + \varepsilon_{23,13}\right) = 0;$$

$$\boxed{i = 1, j = 3} \Rightarrow \boxed{5'} = \boxed{5};$$

$$\boxed{i = 2, j = 3} \Rightarrow \boxed{6'} = \boxed{6}.$$

Compatibility Equations

- Further reductions are possible. Only 3 out of the 81 are independent.

$$\frac{\partial^2}{\partial x^2}\boxed{1}+\frac{\partial^2}{\partial y^2}\boxed{2}-\frac{\partial^2}{\partial z^2}\boxed{3}=\frac{\partial^2}{\partial x\partial y}\boxed{4}:\ \frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial\varepsilon_{xy}}{\partial z}+\frac{\partial\varepsilon_{yz}}{\partial x}+\frac{\partial\varepsilon_{zx}}{\partial y}\right);$$

$$\frac{\partial^2}{\partial y^2}\boxed{2}+\frac{\partial^2}{\partial z^2}\boxed{3}-\frac{\partial^2}{\partial x^2}\boxed{1}=\frac{\partial^2}{\partial y\partial z}\boxed{5}:\ \frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial\varepsilon_{yz}}{\partial x}+\frac{\partial\varepsilon_{zx}}{\partial y}+\frac{\partial\varepsilon_{xy}}{\partial z}\right);$$

$$\frac{\partial^2}{\partial z^2}\boxed{3}+\frac{\partial^2}{\partial x^2}\boxed{1}-\frac{\partial^2}{\partial y^2}\boxed{2}=\frac{\partial^2}{\partial z\partial x}\boxed{6}:\ \frac{\partial^4\varepsilon_y}{\partial z^2\partial x^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial\varepsilon_{zx}}{\partial y}+\frac{\partial\varepsilon_{xy}}{\partial z}+\frac{\partial\varepsilon_{yz}}{\partial x}\right).$$

Outline

- Generalized Displacement
- Small Deformation Theory
- Continuum Motion & Deformation
- Strain & Rotation
- Principal Strains
- Spherical and Deviatoric Strain
- Cylindrical Strain and Rotation
- Spherical Strain and Rotation
- Strain Compatibility
- Domain Connectivity